电气
竞调

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线性代数

1. 逆矩阵 $|A| \neq 0$ · 且A可逆则 $A^{-1} = \frac{1}{|A|} A^*$ 逆矩阵求法为伴随矩阵

A^* 为伴随矩阵

$$A = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & & & \\ \vdots & & & \\ A_{1n} & & & A_{nn} \end{pmatrix}$$

A_{ij} 为余子式.

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. $\det(A^{-1}) = [\det(A)]^{-1}$

4. 高斯消去法

增广矩阵 → 上三角矩阵 → 自下而上

$\underbrace{\quad}_{\text{先消第一列，再消第二列}}$ ....

Exp: $\begin{pmatrix} 1 & -2 & 3 & 3 \\ 0 & 5 & -7 & 3 \\ 0 & 0 & 0 & 7 \end{pmatrix}$ 无解

$\begin{pmatrix} 1 & -2 & 3 & 3 \\ 0 & 5 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 多解 $(x, y, z) = \left(\frac{1}{5}(-t+21), \frac{1}{5}(7t+3), t\right)$

5. 齐次行列式 非齐次行列式

x_g $x_g + x_p$

通解 通解 + 特解

背 6. 若 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 则 $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ 伴随

若 $AX = B$

则 $X = A^{-1}B$.

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7. vector space 向量空间

若 any two vectors in a space. is closed under addition and multiplication then this vector in closed. (such as R^2)

For the line $y=2x$, itself is closed under addition and multiplication, so it is a subspace of R^2.

☆ Because a vector space must be closed under scalar multiplication, each vector space must contain the zero vector

(如 $y=2x+1$ 不包含 0 向量，可以验证它不是一个 subspace)

(如 $x\geq 0, y\geq 0$ 也不是子空间 · isn't closed under scalar)

✓ nullspace 零空间

The solution set of the linear system $Ax=0$ is called the nullspace of A

✓ trivial subspace 平凡子空间

0 零子空间 V 全空间 都称为平凡子空间

8. 线性组合 linear combination

若 $k_1v_1+k_2v_2+\dots+k_mv_m=0$ 当且仅当 $k_i=0$ 则说明 v_i 线性无关

The set of all linear combinations of the vectors v_i is called their span.

如 $\hat{i}=(1,0,0)$ 与 $\hat{j}=(0,1,0)$ 的 span 为 x-y plate

若 $v_2=2v_1+3v_3$

$\text{span}\{v_1,v_2,v_3\}=\text{span}\{v_1,v_2\}=\text{span}\{v_1,v_3\}=\text{span}\{v_2,v_3\}$

☆ 空间的基是能表示空间的最少向量组，数量表示空间的维度

basis independent dimension

eg: Consider the vectors:

$v_1=(1,-2,3)$, $v_2=(-1,-4,9)$ and $v_3=(-1,0,1)$

(a) describe the set span $\{v_1,v_2,v_3\}$

(b) Find a basis for P

(c) what is dim P

span 怎么求

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(a) $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$$(k_1, k_2, k_3)\begin{pmatrix} 1 & -1 & -1 \\ -2 & -4 & 0 \\ 3 & 9 & 1 \end{pmatrix} = 0 \qquad kV = 0$$

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -2 & -4 & 0 & 0 \\ 3 & 9 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

infinite solutions for k. so they are dependent

eliminate V_3.

and a vector nomal to the plane is the cross product

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ +1 & -2 & 3 \\ -1 & -4 & 9 \end{vmatrix} = -6\hat{i} - 12\hat{j} - 6\hat{k} = -6(\hat{i} + 2\hat{j} + \hat{k})$$

take $\vec{N} = (1, 2, 1)$ as a normal vector of P. and the span is

$x + 2y + z = 0$

(b.) $\{v_1, v_2\}$. or $\{v_1, v_3\}$ or $\{v_2, v_3\}$ or $\{v_1, v_2, v_3\}$

(c) 2

9. 矩阵的秩、行空间、列空间.

eg: Compute its rank and find a basis for its row space. and a basis for its column space.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 2 & 0 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

rank(A) = 2

basis for RS(A) (1,1,1) (0,2,0)

To find the column basis. we try to point at the basis for A^T

$$A^T = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

basis for CS(A)

~~(1, -1, 2) (0, 2, -2)~~

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

竖着写

$|A| = |A^T|$ $|AB| = |A||B|$

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10. determinants 行列式   only for square matrix

(1) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\det A = (+1)a_{11}a_{22} + (-1)a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}$

for 3×3 A

$|A| = \cancel{\Sigma(\mathrm{sgn}\,\sigma) a_{ijk} \cdots} \ \Sigma(\mathrm{sgn}\,\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

not practical ×

(2) 行列式变换   用行变换 (only ?)

① 两行交换，行列式加负号   ⑦ $|A| = |A^T|$

② 行 × 某数，行列式 × 某数

③ 某行 + a row is multiple → not change

④ 三角行列式等于对角元素乘积

⑤ 若某行为零，则行列式为零

⑥ $|AB| = |A||B|$

eg: compute the determinant

$A = \begin{pmatrix} 4 & 4 & -1 \\ 2 & -3 & 0 \\ -1 & 2 & -1 \end{pmatrix}$

$\to \det A = -\det \begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 0 \\ 4 & 4 & -1 \end{pmatrix}$ 尽量让小的在上面好算.

$= -\begin{vmatrix} -1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 19 \end{vmatrix} = +19$

eg: A is a 3x3 matrix $\quad \det A = 5 \quad \det(2A) = ?$

answer = 40

(3) Laplace expansions 拉普拉斯展开 (余子式)

$$\det A = \sum_{j=1}^{n} a_{ij}\, cof(a_{ij})$$

$$cof(a_{ij}) = (-1)^{i+j}(a_{ij}minor)$$

eg: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det A = a_{11}\left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right) + a_{12}(-1)\left(\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right) + a_{13}\left(\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right)$$

11. Adjugate Matrix 伴随矩阵

$$Adj = [cof(a_{ij})]^T$$

eg: $\det A = k \quad \det(Adj\,A) = ?$ $\quad$ A为 $n \times n$ matrix

$A^{-1} = \frac{1}{|A|}A^* \quad A^{-1} = \frac{1}{k}A^* \quad \therefore A^* = kA^{-1}$

$|A^*| = |kA^{-1}| = k^n|A^{-1}| = \frac{k^n}{k} = k^{n-1}$ $\quad \det(A^{-1}) = [\det A]^{-1}$

12. Cramer's Rule 克莱姆法则 $\quad \boxed{Ax=b \text{ and then } x_j = \frac{\det A_j}{\det A}}$

$A_j \rightarrow$ the matrix formed by replacing column j of A by the column vector b

eg: Find the value of y. if

$$\begin{cases} 2x - 3y + 4z = 1 \\ 4x + 9y = 0 \\ 7x - 2y + 5z = 0 \end{cases}$$

求y 所以第2列替换

$A_2 = \begin{pmatrix} 2 & 1 & 4 \\ 4 & 0 & 0 \\ 7 & 0 & 5 \end{pmatrix} \leftarrow$ Augment matrix $= \begin{pmatrix} 2 & -3 & 4 & 1 \\ 4 & 9 & 0 & 0 \\ 7 & -2 & 5 & 0 \end{pmatrix}$

$$y = \frac{\begin{vmatrix} 2 & 1 & 4 \\ 4 & 0 & 0 \\ 7 & 0 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 4 \\ 4 & 9 & 0 \\ 7 & -2 & 5 \end{vmatrix}} = \frac{10}{67}$$

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13. 线性变换 linear transformation

满足 function $T: V \to W$ 后 $V=W$ 的 T 叫作线性变换

$T(x_1+x_2)=T(x_1)+T(x_2)$ $T(kx)=kT(x)$

△若某变换不能将 $0 \to 0$ 则一定不为线性变换 ☆

若 R^n 的基失为 $B=\{b_1, b_2 \cdots b_n\}$. 则 R^n 中的向量 x. 表示为

$x = k_1b_1 + k_2b_2 + \cdots + k_nb_n$

那么 $\vec{k} = (k_1, k_2 \cdots k_n)$ 叫作 x 在 B 基失下的分向量.

$T(x) = k_1T(b_1) + k_2T(b_2) + \cdots + k_nT(b_n)$

eg: If $T: R^2 \to R^2$ is the linear transformation that maps (1,1) to (3,4) and (-1,1) to (2,2), what's the imge of (3,1)?

基失就2个

since (1,1). (-1,1) independent and span R^2. They form the basis of R^2.

$$\begin{cases} k_1 + (-k_2) = 3 \\ k_1 + k_2 = 1 \end{cases} \quad k_1 = 2 \quad k_2 = -1$$

$(3,1) \to (6,8) + (-2,-2) = (4,6)$

eg: If $T: R^3 \to R^3$ is the linear operator that maps (1,0,0) to (1,2,3), (0,1,0) to (-1,1,1) and (0,0,1) to (1,-2,0) what's the inge of (3,4,-5)

standard matrix.

$$(3,4,-5) \to \cancel{(3,4,-5)} \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 20 \\ 13 \end{pmatrix}$$

14. Nullity Theorem 退化

nullity (T) + rank (T) = n = dim (domain T)

when T is one-to-one. the nullity is zero

ker T = $\{x : T(x) = 0\}$ the dimension of ker T is called nullity

解空间

kernel of T is a subspace of the domain $R^n \to R^m$

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eg: determine the kernel, nullity, range and rank of the $T: R^3 \to R^2$ given by the equation $T(x) = Ax$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

化为阶梯阵 $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix}$

$-6z + -3y = 0 \quad \therefore \; y = -2z \qquad x = z$

$\ker T: \{x : T(x) = 0\} = \{(1,-2,1)t : t \in R\}$

nullity $(T) = 1$

$\text{rank}(T) = 3 - 1 = 2$

15. eigenvalues and eigenvectors 特征根和特征向量. only for square matrix

✓ if $Ax = \lambda x$ 则称 λ 为 A 的特征根. x 称为特征向量

$\det(A - I\lambda) = 0$.

eg: Find the eigenvalues and eigenvectors of the matrix:

(1) $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{pmatrix}$ $\quad \det(A - \lambda I) = (1-\lambda)(4-\lambda) + 2$

$= \lambda^2 - 5\lambda + 6$.

$\lambda = 2, 3$.

(2) $(A - \lambda I)x = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} x = 0$. $\qquad (A - \lambda I)x = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} = 0$

$x = k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\qquad x = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

✓ if a square matrix A has zero as an eigenvalue, then $\det A = 0$ and it is not invertible.

✓ characteristic polynomial 特征多项式

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16. eigenspaces 特征空间

eg: Find a basis for the corresponding eigenspace

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x = \begin{pmatrix} t_1 - t_2 \\ t_1 \\ t_2 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\{(1,1,0)^T, (-1,0,1)^T\}$ is the basis for $E_{\lambda=0}(A)$

定义: $E_\lambda(A) = \{x : Ax = \lambda x\}$

17. Cayley-hamilton 凯利-哈密尔顿

如果 $P(\lambda) = \det(A-\lambda I)$ 为 A的特征多项式. 则 $P(A)=0$

eg: $A = \begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}$

$$A-\lambda I = \begin{pmatrix} 4-\lambda & -2 \\ 5 & -3-\lambda \end{pmatrix} \quad \begin{vmatrix} 4-\lambda & -2 \\ 5 & -3-\lambda \end{vmatrix} = -(4-\lambda)(3+\lambda)+10$$

$$= (\lambda-4)(\lambda+3)+10$$

$$= \lambda^2 - \lambda - 2$$

so $A^2 - A - 2I = 0$

18. Nullspace.

$$\begin{cases} x - 2y + 3z = 0 \\ 2x + y - z = 0 \\ -3x - 4y + 5z = 0 \end{cases}$$

the solution is $\{(-\frac{1}{5}t, \frac{7}{5}t, t) : t \in R\}$

This is the nullspace of the cofficient matrix:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ -3 & -4 & 5 \end{pmatrix}$$

19. T是一个 $R^n \to R^m$ 的线性变换. 使得 $T(x)=0$ 的向量空间x称为T的kernel.
ker(T) 是定义域 R^n 的子空间, 它的维数称为 nullity. T的所有象集叫作T的range
记作 $R(T) = \{T(x) : x \in R^n\}$ range of T是 R^m 的子空间, 它的维数称为 rank

$$\text{nullity}(T) + \text{rank}(T) = n = \text{T的定义域维数}.$$

可知 $T: R^n \to R^n$ 的 nullity 为零. rank为n.

20. 线性变换及其矩阵表示

$$Tx = Ax$$

A为基矢的像作为列向量的变换矩阵

将x在基矢 $\{b_1, b_2 \cdots b_n\}$ 下的表示写为列向量 $(k_1, k_2, \cdots k_n)^T$

则 x在T下的像由 $A\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$ 确定.

线性变换的矩阵表示

eg: If $T: R^2 \to R^2$ is the linear transformation that maps (1,1) to (3,4). (-1,1) to (2,2) what's the image of (3,1)

the transform matrix: $\begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}$

(3,1) under basis of (1,1), (-1,1) is $(2,-1)^T$

thus the image is $\begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

一定写作列向量!

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数论

1. $\frac{b}{a}$ 写作 $a|b$ a divides b (b is divided by a)

2. 被4整除 = 后两位被4整除
   被5整除 = 个位为0或5
   被8整除 = 后三位被8整除
   被3或9整除 = 数位和被3或9整除

3. relatively prime 互质
   if a and b is relatively prime, and c can be divided by a and b. then c can be divided by ab.

4. $b = qa + r$ q为商 r为余数.

5. 质数 prime
   $\sum_{k=1}^{n} \frac{1}{p_k}$ p_k为第k个质数. 该数列发散.
   $\pi(x)$表示被x小的质数个数 $\pi(x) \approx x/\log x$

6. gcd 最大公约数   lcm 最小公倍数.
   $\gcd(a,b) \cdot \mathrm{lcm}(a,b) = ab$

7. 求最大公约数的方法 The Euclidean algorithm
   compute $[2002, 315]$
   $2002 = 6 \cdot 315 + 112$
   $315 = 2 \cdot 112 + 91$
   $112 = 91 \cdot 1 + 21$
   $91 = 21 \times 4 + 7$
   $21 = 7 \times 3 + 0$ → stop signal
   最后一个除数.
   $[2002, 315] = 7$

8. The Diophantine Equation $ax + by = c$
   (1) if c couldn't be divided by $\gcd(a,b)$, there is no solution
   (2) if $[a,b] = d$, then could find
   $a x_0 + b y_0 = d$

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eg: Find the solution of $15x+49y=8$

$[15,49]=1$

$15(49t+a)+49(b-15t)=8$

$15a+49b=8$

△ how to find the primary solution

$49=3\cdot15+4$

$15=4\times3+3$

$4=3\times1+1 \quad \to \quad 1=4-3\times1=4-(15-4\times3)\times1$

$3=1\times3+0$

$=4\cdot4-15$

$=4\cdot(49-3\cdot15)-15$

$=4\cdot49-13\cdot15$

$1=15(-13)+49\cdot4 \xrightarrow{\times 8} 8=15(-104)+49\cdot32$

$\begin{cases} x=49t-104 \\ y=32-15t \end{cases}$

9. 同余

① $a\equiv b \pmod n$ 则 $a+c\equiv b+c \pmod n$ $\quad ac\equiv bc \pmod n$

② $a_1\equiv b_1 \pmod n$ $a_2\equiv b_2 \pmod n$ 则 $a_1\pm a_2\equiv b_1\pm b_2 \pmod n$

$a_1a_2\equiv b_1b_2 \pmod n$

③ 若 $ab\equiv ac \pmod n$

则 $b\equiv c \pmod n$ if $\gcd(a,n)=1$

$b\equiv c \pmod{\frac{n}{d}}$ if $d=\gcd(a,n)>1$

④ Fermat's Little theorem

$a^{p-1}\equiv1 \pmod p$ $\quad$ p为质数. 且 $[a,p]=1$

eg: What's the remainder when 2^{345} is divided by 29?

$2^{345}\equiv2^{28\times11+9}\equiv2^9\equiv2^5\cdot2^4\equiv48 \pmod{29}$

so the remainder is 19

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10. The Congruence equation $ax \equiv b \pmod{n}$
This equation has a solution if and only if $d = \gcd(a, n)$ divides b.

eg: Solve the congruence equation $18x \equiv 5 \pmod{7}$

$18x \equiv 12 \pmod{7}$

$\xrightarrow{[6,7]=1} 3x \equiv 2 \pmod{7}$

$\longrightarrow 3x \equiv 9 \pmod{7}$

$\xrightarrow{[3,7]=1} x \equiv 3 \pmod{7}$

$x = 7t + 3$

eg: There's only one integer x, between 100 and 200 such that $144x \equiv 22 \pmod{71}$

$144x \equiv 93 \pmod{71}$

$\xrightarrow{[3,71]=1} 48x \equiv 31 \pmod{71}$

$\xrightarrow{[2,71]=1} 48x \equiv 102 \pmod{71}$

$\longrightarrow 24x \equiv 51 \pmod{71} \longrightarrow 24x \equiv 122 \pmod{71} \rightarrow 12x \equiv 61 \pmod{71}$

$\rightarrow 12x \equiv 132 \pmod{71} \longrightarrow x \equiv 11 \pmod{71}$

$x = 71 \times 2 + 11 = 142 + 11 = 153$

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抽象代数. ABSTRACT Algebra

1. $\mathbb{Z}$ 整数集 $\mathbb{Q}$ 有理数集 $\mathbb{R}$ 实数集

$\mathbb{C}$ 复数集

$M_{m\times n}(S)$ 所有m×n矩阵,(元素在集S中)

$M_n(S)$ ……n×n方阵, ———.

$\mathbb{Z}^+$ 正整数集 $\mathbb{Q}^+$ 正有理数集 $\mathbb{Z}^+$ doesn't contain 0.

2. function $f: S\times S \to S$

every ordered pair of elements of S, the result is also in S, it is called a binary operation on S

Let $S=\mathbb{Z}$ $f(a,b)=a+b$. Addition is a binary operation

while substraction is not a binary operation on $\mathbb{Z}^+$

3. Binary structure

A nonempty set S, together with a binary operation defined on it, is called a binary structure. $(\mathbb{Z},+)$ and $(\mathbb{Z},-)$ are binary structure while $(\mathbb{Z}^+,-)$ is not

✓ Semigroup 半群

a binary operation, $*$, on S is said to be associative: (满足结合律)

$(a*b)*c = a*(b*c)$

a binary structure whose operation is associative is called a semigroup.

$(\mathbb{Z},+)$ ✓ $(\mathbb{Z},-)$ ×

✓ Identity

$(S,*)$, then an element e of S, satisfy

$a*e = e*a = a$

then e is called the identity.

if $(S,*)$ has an identity, then its unique

✓ Monoid 幺半群

a semigroup that has an identity is called a monoid

$(\mathbb{Z},+)$ is a monoid. Its identity is zero. $a+0=0+a$

binary structure $\Rightarrow a*(b*c) = (a*b)*c \rightarrow$ semigroup
$\rightarrow a*e = e*a = a \rightarrow$ monoid
$\rightarrow a*\tilde{a} = \tilde{a}*a = e \rightarrow$ group

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✓ 群 Group

$(S,*)$ is a monoid, $a, \tilde{a}$ are elements in S

$a*\tilde{a} = \tilde{a}*a = e$

Where e is the identity $\rightarrow \tilde{a}$ is the inverse of a.

△ A monoid with the property that every element in S that has an inverse is called group.

$(\mathbb{Z}, +)$ ✓ $(\mathbb{Z}, \times)$ ×

★ If $(G,*)$ is a group. then for any elements a and b in G the linear equation $a*x=b$ can be solved and the solution is unique as well as $y*a=b$

上述结论对半群、幺半群不成立

✓ Commutative / Abelian Group 阿贝尔群

$(S,*)$ is a binary structure $a*b=b*a$ holds for every two elements in S, then the binary operation $*$ is commutative

一个半群,幺半群,群 whose binary operation is commutative is called Abelian

✓ Finite / infinite 有限群和无限群

If the set S contains precisely n elements, then it is finite with the order of n.

✓ General Linear group

$GL(n,\mathbb{R})$ is the subset of $M_n(\mathbb{R})$ R域所有可逆方阵构成的集合

R域也可改为C、Q 但不可为Z域.

✓ $SL(n,\mathbb{R})$ is the subset of $GL(n,\mathbb{R})$ which whose 行列式为1

可被延拓到 Z、C、Q 域.

— Special Linear group

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✓ $U_n = \{z : z^n = 1\}$ consisting of the n^{th} roots of unity

$(U_n, *)$ is a finite group

✓ 对称群 symmeteric group. 最小的非阿贝尔对称群的阶数为6

$X \to X$ with n letters → the order is $n!$

For symmtric group, the smallest order for an non-Abelian group is 6.

✓ 二面体群 Dihedral group.

The order of D_n is $2n$. and D_3. S_3 is the same group.

✓ Multiplicable group.

In any group table, every element of its underlying set appears once, and only once, in every row and every column in the body of the table. If this criterion is not satisfied by a multiplication table, it cannot be a group table.

✓ Cyclic group 循环群

$G = \{a^n = n = 0, 1, 2 \cdots\}$

the element a is a generator of the group and a^0 is the identity.

✓ Additive group of integers modulo n 加模循环群

$(Z_n, \oplus)$ $(a+b) \bmod n$

The integer m is a generator of $(Z_n, \oplus)$ if and only if m, n 互质

△ Let G be a cyclic group with generator a. and let n be the smallest integer such that $a^n = e$:

The element a^m is a generator of G if and only if m, n 互质

eg: U_4 is generated by i. and is also generated by $-i = i^3$. $[3, 4] = 1$

Every cyclic group is an Abelian

✓ Subgroup 子群

Let $(G, *)$ be a group. If there's a subset H of G such that $(H, *)$ is also a group, then we say that H is a subgroup of G. and write $H \leq G$

If $H < G$, then H is a proper subgroup 真子群 of G.

Trivial group: $\{e\}$ and G itself.

No.
Date.

✓ Klein four-group / viergruppe

最小的非循环群
除单位元外.其余因素阶为2,同构于4阶二面群

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

An Abelian not a cyclic group
has three proper, nontrivial group
$\{e,a\}$ $\{e,b\}$. $\{e,c\}$

To make a subgroup
(1) H is closed under operation
(2) contains the identity
(3) the inverse of every element in H is also in H

$\mathbb{Z}_n = \{0,1,2,\dots,n-1\}$

determine the order of an element in a cyclic group:

The order of $a \in G$ is the smallest positive integer n such that $a^n = e$

eg: In $\mathbb{Z}_6$. the order of element 4 is 3.
$4+4 \equiv 2 \pmod 6$ $\quad 2+4 \equiv 0 \pmod 6$.

eg: In the Klein four-group the order of every element is 2

eg: In the additive group $\mathbb{Z}$, every element m expect the identity has infinite ~~group~~ order.

✓ Lagrang's theorem 拉格朗日定理

Let G be a finite group. If H is a subgroup of G, then the order of H divides the order of G.

→ If G is a finite group and of order n, and m divides n, G must have a subgroup of order m. → It's true for Abelian, but not others.

阿贝尔有限群的子群order为n的约数 (循环群必为阿贝尔群)

→ If G be a finite, cyclic group. Then G has exactly one subgroup — a cyclic subgroup — of order d for every (positive) divisor d of n. If G is generated by a, then the subgroup generated by the element $b = a^m$ has order $d = n/\gcd(m,n)$

有且仅有一个

有限循环群的子群order为n的约数. 且由 a^m generate 的子群 order都为
$d = n/\gcd(m,n)$

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有限群G的阶数 — 集合G的元素个数

元素a的阶数 → 使 $a^r=e$ 成立的最小整数r

若G为有限循环群，a为其生成元，则若G的阶数为n，a的阶数也为n

若G为无限循环群，则它的生成元为 a, a^{-1}

eg：求 $\langle Z_{12}, \oplus \rangle$ 的生成元，子群

生成元为与12互质的数 1, 5, 7, 11

子群 (0), (1), (2), (3), (4), (6)

求 $G=\langle a^2 \rangle$ 为12阶群的生成元

$a^2 \cdot a^{10} \cdot a^{14} \cdot a^{22}$ (1, 5, 7, 11)

✓ Cauchy's theorem 柯西定理

G为order为n的有限群，p为可整除n的素数，则G至少有1个阶为p的子群

柯西定理和拉格朗日定理适用于任意有限群

✓ Sylow's first theorem 西洛定理

G为阶数为n的有限群． $n=p^k \cdot m$，p为一个不可整除m的质数，则G对于每个不同的k至少有一个阶为 p^k 的子群

eg：对于G（阶为48） $48=2^4 \cdot 3$

至少有阶为 $2^0, 2^1, 2^2, 2^3, 2^4, 3^1$ 的子群各一个

✓ Isomorphism 同构性

G_1, G_2 为同构性群： $G_1 \cong G_2$

如果 G_1 有某个结构性性质不与 G_2 共有，则 G_1, G_2 非同构

任意阶为3的群都与 Z_3 同构

✓ Direct Product 直乘

$(G_1 \times G_2, *)$ is called the direct product of $G_1 \cdot G_2$.

对称群的阶为n!

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若 G_1 阶为m. G_2 阶为n. 则 $G_1 \times G_2$ 阶为mn

如果 G_1, G_2 均为阿贝尔群，则用 $G_1 \oplus G_2$ 代替 $G_1 \times G_2$，称为direct sum

$Z_2 \oplus Z_3$ is cyclic and is generated by $(1,1)$

$Z_2 \oplus Z_4$ is not cyclic

$\Rightarrow$ The difference is $(2,3)=1$ while 2,4 不互质

若m,n互质则 $Z_m \oplus Z_n$ 循环且与 Z_{mn} 同构

eg $Z_8 \oplus Z_3 \oplus Z_{25}$ is cyclic and isomorphic to Z_{600}.

eg: Find the order of the element $(1,6,25)$ in the group $Z_2 \oplus Z_{10} \oplus Z_{30}$

$m=(2,5,6)=30$

An Abelian group can be represented

$Z_{(p_1)^{k_1}} \oplus Z_{(p_2)^{k_2}} \cdots \oplus Z_{(p_r)^{k_r}}$ elementary divisors p_i为素数. 可以相同

$Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_t}$ invariant factors $m_i \mid m_{i+1}$

✓ Group Homomorphism 群同态

G_1, G_2 存在函数 $f: G_1 \to G_2$ $(G_1, \cdot)$ $(G_2, *)$

$f(\underset{\uparrow}{a \cdot b}) = f(a) \underset{\uparrow}{*} f(b)$ $(a,b) \in G_1$

G_1中运算 G_2中运算 单一

—— 映射的群同态称为 monomorphism 满同态

onto epimorphism 满射

if it is one-to-one and onto : isomorphism 同构

即同构是同态的特例

A homomorphism from a group to itself = endomorphism 自同态

An isomorphism from a group to itself : automorphism 自同构

性质: ① e is identity in G_1, then $f(e)$ is indentity in G_2

② $g \in G_1$ has finite order m, then $f(g)$ has order m in G_2

③ $a^{-1} \in G_1$ is the inverse of a in G_1, then $f(a^{-1})$ is the inverse of $f(a)$ in G_2

homomorphism 同态 → one-to-one → monomorphism 单一同态 (kernel trival) → isomorphism 同构
homomorphism 同态 → onto → epimorphism 满同态 → isomorphism 同构

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④ H is a subgroup of G_1, $f(H)$ is a subgroup of G_2
⑤ 若 G_1 有限 $f(G_1)$ 的阶整除 G_1 的阶
- G_2 - - $f(G_1)$ 的阶 — G_2 - - -
⑥ H' is a subgroup of G_2, $f^{-1}(H')$ is a subgroup of G_1
f 的 kernel 为 g ($g \in G_1$) 满足 $f(g) = e'$ (e' 为 G_2 的 identity)
一个同态 (homomorphism) is a monomorphism 当且仅当 kernel 平凡
求出 G_1 中所有满足 $f(g) = e'$ 的 g

P40 群同态的性质☆
$(a*b)^{-1} = b^{-1}*a^{-1}$

? what's the meaning of 'onto' - 满射

△ $(R,+)$ is isomorphic

△ An infinite group can be isomorphic to one of its proper subgroups
同构的条件：同态. 双射

? 如何证明 onto.
$\phi_a(g) = aga^{-1}$
取 $aga^{-1} = j$ $g = a^{-1}ja$ $\phi_a(g) = j$. 因此 onto. 取反函数

Ring 环
A set R, together with <u>two binary operations</u> is called a ring
(+, · for example)
satisfy the three following: ① $(R,+)$ is abelian $a+b=b+a$
② $(R,\cdot)$ is semigroup $(a\cdot b)\cdot c = a\cdot(b\cdot c)$
③ $a\cdot(b+c) = a\cdot b + a\cdot c$ and $(a+b)\cdot c = a\cdot c + b\cdot c$
进一步 若 $(R,\cdot)$ is a monid. say, has ~~an~~ identity, then the ring is called a ring with unity, and we require the identity of $(R,+)$ is $\boxed{\text{distinct with}}$ the identity of $(R,\cdot)$
更进一步. 若 $(R,\cdot)$ is commutative, say $a\cdot b = b\cdot a$, the ring is called the commutative ring

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$(S,+,\cdot)$ is a subring of $(R,+,\cdot)$

Let R be a ring. The smallest positive integer n such that $na=0$ for every a in R is called the characteristic of the ring, and we write char $R=n$. If no such n exists, we say that R has characteristic zero.

eg $(\mathbb{Z},+,\cdot)$ is called the ring of integers, $n\mathbb{Z}$ is the subring of $\mathbb{Z}$, while $n>1$, $n\mathbb{Z}$ is not a ring with unity.

eg: $(\mathbb{Z},+,\cdot)$ is subring of $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$ and $(\mathbb{C},+,\cdot)$

eg: $(\mathbb{Z}_n,+,\cdot)$ is the ring of integers modulo n.

eg: $(M_n(\mathbb{R}),+,\times)$ is a ring with unity, a noncommutative ring if $n>1$

eg: the set $C=\{a+b\sqrt[3]{2} : a,b\in\mathbb{Z}\}$ does not form a ring. since it's not closed under multiplication:

$$(a_1+b_1\sqrt[3]{2})(a_2+b_2\sqrt[3]{2}) = a_1a_2+(a_1b_2+a_2b_1)\sqrt[3]{2}+b_1b_2\sqrt[3]{4}$$

eg: with the operation of $+,\cdot$ in $\mathbb{C}$. the set

$\mathbb{Z}[i]=\{a+bi : a,b\in\mathbb{Z}$ and $i=\sqrt{-1}\}$ is a subring of $\mathbb{C}$.

namely, the ring of Gaussin integers

eg: 考虑 R 中所有多项式的collection

$R[x]=\{r_0+r_1x+r_2x^2+\cdots+r_nx^n : r_i\in R\}$ is a ring

namly, the ring of polynomials in x over R

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函数的结合律,交换律

$(f+g)x=f(x)+g(x)$ $(fg)(x)=f(x)g(x)$

eg: R是a ring with unity. S是它的子集. 它的unity not neccessary the same as R's

Attention: 矩阵的乘法 is not commutative ⇒ 可用作反例

However $\phi(A)=\det(A)$ → group homomorphism

✓ Ring Homomorphisms 环同态 whether preserve addition. multiplication

Let $(R,+,\times)$ and $(R',\oplus,\otimes)$ be rings, and if a function $f: R\to R'$

$f(a+b)=f(a)\oplus f(b)$

$f(a\times b)=f(a)\otimes f(b)$

f is called a ring homomorphism

性质

1. The kernel of a ring homomorphism is the set $\ker f=\{a\in R: f(a)=0'\}$ $0'$ is the identity of the addition in R'

the kernel of $f: R\to R'$. is always a subring of R.

2. The image of R, $\phi(R)=\{\phi(r): r\in R\}$ is a subring of R'

3. $\phi(-r)=-\phi(r)$

4. The image of 0, the additive identity in R, must be $0'$. the additive identity in R'

eg: $f_p: R\to R$ given by $f(a)=a^p$ is a ring homomorphism

eg. Let $(R,+,\times)$ and $(R',\oplus,\otimes)$ be rings, it's possible for $(R,+),(R',\oplus)$ be a group 同态 $(R,\times),(R',\otimes)$

but not a ring 同态

Z_6 的 identity 为 0.
unity 为 1.
unit 为 1, 5 ($1\times1\equiv1$, $5\times5\equiv1$)

☆ for any rings R and R'. the zero map, $z: R\to R'$ given by $z(r)=0'$ for every r in R, is always a ring homomorphism
from the equation
$a\cdot(b+c) = a\cdot b + a\cdot c$ let $c=0$. (the additive identity)
$a\cdot(b+0) = a\cdot b + a\cdot 0 = a\cdot b$ that's to say $a\cdot 0=0$

✓ Integral Domains 整环
If n is prime, then Z_n is an integral domain.
a, b are both nonzero elements of ring. but product ab equal to 0.
$a\neq0$ and $ab=ac \Rightarrow b=c$
is only guaranteed when there are no zero divisors.

A nonzero element $m\in Z_n$ is a zero divisor if and only if m, n 互质
整环条件: ① 乘法适合交换律 $ab=ba$
② 有单位元 e ☆ $a*e=e*a=a$
③ 没有零因子 $ab=0$ 可得 $a=0$ 或 $b=0$
若一个元素既是左零因子又是右零因子, 则称它为零因子 (两个非零元相乘为零)
所有的域都是整环
零元 ≠ 零因子.
本身是零元 若存在非零元 b, $ab=0$, 则都为零因子.
单位 (unity) ≠ 单位元 (identity)
unit ≠ unity. unit is a nonzero element with a multiplicative inverse
△ 若 every 非零元都有倒数(在R中), 则称 R 为 a division ring
△ a commutative division ring is called a field
✓ Field 域
a commutative division ring → field
$a\cdot b=b\cdot a$ for $a\neq0$ a always have the inverse a^{-1}

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抽象代数总结.

1. binary operation　加法、乘法封闭．则称为 binary structure

→ 再有 $a*(b*c)=(a*b)*c$ 结合律　称为半群

→ 再有 $a*e=e*a=a$ 对任意 $a\in S$ 成立，e 叫 identity．S 叫 monoid

→ 再有 $a*\bar{a}=\bar{a}*a=e$ 对任意 a invertible　称 S 为 group．

$(R,+)$ 的 identity 为 0．　$(R,\times)$ 的 identity 为 1．

not neccessary the same. sometimes. needs to be distinct

→ 再有 $a*b=b*a$　交换律　叫作 Abelian 阿贝尔群．

任意循环群均为阿贝尔群

2 ① $GL(n,R)$ 是所有 R 域可逆方阵的 collection，

② $SL(n,R)$ 是 $GL(n,R)$ 中行列式为 1 的子群．

③ $U_n=\{z: z^n=1\}$　order 为 n

④ 对称群 symmetric group　$S_n=\{1,2,\dots n\}$　order $=n!$ 最小的非阿贝尔对称群为 $n=3$，当 $n\geq 3$ 时对称群就是非阿贝尔的

⑤ 二面体群 Dihedral group　D_n　order 为 $2n$．

$n=3$ 时　二面体群和对称群为同一个群

最小的阿贝尔非循环群 V_4．除单位元外，其余元素阶均为 2，同构于 4 阶二面体群 $n=2$

⑥ 模 n 群　　$Z_n=\{0,1,2,\dots n-1\}$

	$(Z_n,\oplus)$	$(Z_n,\otimes)$
	加模 n 群	乘模 n 群
identity	0	1

3. 循环群　cyclic group

$G=\{a^n: n=0,1,2\dots\}$　a is the generator (生成元)

n 表示循环几次．并不表示幂．

G 的生成元为 a^m　(m 与 n 互质) if and only if　即 G 的生成元仅可能为 a^m．$(m,n)=1$

循环群必为阿贝尔群　(阿贝尔群不一定为循环群)

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4. 子群

平凡子群 {e} and itself

子群条件：① 运算封闭.

② 有单位元

③ 逆元在子群中.

✓循环群子群

使得 $a^n=e$ 的 n 为循环群的阶数

✓拉格朗日定理

有限群 G 的子群阶数 m，整除 G 的阶数 n，即 $m|n$

(即对于 G_{12}，G_7 一定不为其子群，但 G_6 不一定为其子群；$m|n$ 为必要非充分条件)

推论1：若 G 为有限阿贝尔群，阶数为 n，则对于 n 的所有因子 d，G 都必有一个阶数为 d 的子群 (至少有)

推论2：若 G 为有限循环群，则 G 有且仅有一个阶数为 d 的子群 (for every divisor of n)。如果 G 生成元为 a，子群生成元为 a^m，则子群阶数为 $d=n/\gcd(m,n)$

✓柯西定理.

G 为 n 阶有限群，p 为 n 的质因子，则 G 有阶为 p 的子群至少一个.

(如 G_{12}，不一定有 G_6 子群，6 不为质数，但一定有 G_2，G_3 子群)

✓西洛定理

G 为 n 阶有限群 $n=p^k m$，$(p,m)=1$ 则 G 的子群中必有阶为 $p^0, p \dots p^k$ 的子群

5. 直积与 $Z_m \oplus Z_n$

① $G_1 \times G_2 = \{(a,b): a\in G_1 \text{ and } b\in G_2\}$ 也写 $G_1 \oplus G_2$

如 $Z_2 \oplus Z_3$ 有 6 个元素 $6=2\times3$

(1,1) 在 $Z_2 \oplus Z_3$ 循环时 如 $3(1,1)=(1,0)$

$3\times1\equiv0 \pmod 3$

$3\times1\equiv1 \pmod 2$

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② $Z_m \oplus Z_n$ 直乘之后循环的充要条件为 $(m,n)=1$
并且 $Z_m \oplus Z_n$ 与 Z_{mn} 同构.
③ 任意有限阿贝尔群均与下面形式的直乘同构.
$Z_{p_1^{k_1}} \oplus Z_{p_2^{k_2}} \oplus \cdots \oplus Z_{p_r^{k_r}}$
order $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ （$p_1, \cdots, p_r$ 不一定不同）.
则 $p_i^{k_i}$ 叫做 G 的 elementary divisors.
或者
$Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_t}$
其中 $m_i \mid m_{i+1}$，m_i 叫做 G 的 invariant factors.

针对阿贝尔群

如 G_{600} 有
$600 = 2^3 \cdot 3 \cdot 5^2$
$3 \times 2 = 6$ 个不同构的阿贝尔群

6. 群同态. $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ $\quad \phi: G \to G'$
同构：同态. 双射 (one-to-one, onto) 原像唯一
monomorphism epimorphism
自同态 endomorphism 自同构 automorphism.
性质 ① e 为 G 中单位元，$\phi(e)$ 为 G' 中单位元.
② $g \in G$ 阶数为 m. $\phi(g) \in G'$ 阶数也为 m
③ a^{-1} 为 a 在 G 中逆元，$\phi(a^{-1})$ 为 $\phi(a)$ 在 G' 中逆元
④ H 为 G 子群，$\phi(H)$ 为 G' 子群
⑤ G 有限. 则 $\phi(G)$ 阶数整除 G 的阶数.
G' 有限，则 $\phi(G)$ 阶数整除 G' 阶数
⑥ H' 为 G' 子群，$\phi^{-1}(H')$ 为 G 子群

如何判断 one to one
$\ker \phi = \{g \in G : \phi(g) = e'\}$ is trivial

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7. 环 $(R,+,\cdot)$

① $(R,+)$ 为阿贝尔群 , identity 0

② $(R,\cdot)$ 为半群 , unity. e

③ 满足分配律 $a\cdot(b+c) = a\cdot b + a\cdot c$

若 $na=0$ for every a in R. n is called the characteristic of the ring, positive integer
we write '$\text{char } R = n$'

eg $R=\{a+b\sqrt{2} : a,b\in\mathbb{Z}\}$ is a ring
$C=\{a+b\sqrt[3]{2} : a,b\in\mathbb{Z}\}$ is not binary operation

8. 环同态

$\phi(a+b)=\phi(a)\oplus\phi(b)$

$\phi(a\times b)=\phi(a)\otimes\phi(b)$

环的核 $\ker\phi = \{a\in R : \phi(a)=0'\}$ $0'$ is the additive identity in R'

① $\phi(R)$ is a subring of R'

② R中的加法单位元0的像也一定为R'中的加法单位元

9. 整环

不含零因子的 commutative ring with unity

若 $ab=0$ 则 $\quad ab=ba \quad ae=ea=a$

a,b中必有一个为0.

10. 域

element a which has an multiplicative inverse is called an unit

every element in R is invertible → a division ring

满足交换律的可分环为域

commutative division ring

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Addition Topic

一. 逻辑学.

$\overline{X+Y} = \overline{X}\cdot\overline{Y}$ $\overline{XY} = \overline{X}+\overline{Y}$

$A+BC = (A+B)(A+C)$ $A+\overline{A}B = A+B$ $AB+\overline{A}C+BC = AB+\overline{A}C$

$AB+A\overline{B}=A$ 事普通的不赘述

二. Set Theory 集合论

equal: $A \subseteq B$. and $B \subseteq A$. $A=B$

equivalent: $A \to B$. 或 $B \to A$ 存在映射函数. $A \approx B$.

equivalent sets have the same cardinality

The cardinal number (or cardinality) of a set is the number of elements in the set.

eg: $A = \{1, 2, \ldots, n\}$ Thus the cardinal number of A is n.

card $A = n$

无限集

如 Z^+. 叫作 countably infinite. 如何一个集合 equivalent to Z^+. 那它就是可数无限.

1. 可数无限. Countably infinite. (card 为 $\aleph_0$)

Z. Q.(有理数). all algebraic number 以及它们的子集 交集 并集 叉乘.

2. 不可数无限 uncountable infinite (card 为 $2^{\aleph_0} = c$)

Q^c $(R-Q)$ 无理数. all transcendental number. 以及 Q^c 的子集

若 $A \to B$. 则 card $A \le$ card B (A中的元素一一映射到B)

若 $B \to A$ 则 card $B \le$ card A (B中的元素一一映射到A)

Power set of A.

consider the family of all A's subset

$$P(A) = \{B : B \subseteq A\}$$

若 A is finite, and has card n. then $P(A)$ has card 2^n.

Algebraic number · 代数数

if x is a root of 整系数多项式方程 $p(x)=0$, then it is an algebraic number.

universal set 全集

complement set 补集

intersection 交集

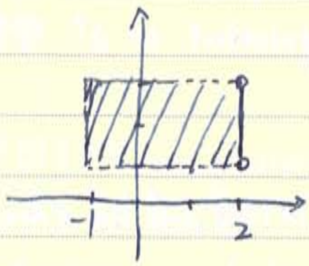
union 并集

symmetric difference 对称差 $A \Delta B = (A-B) \cup (B-A)$

cartesian product 叉乘 $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

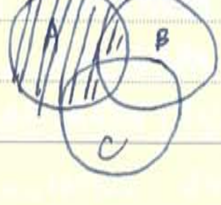
interval 区间

eg: $A = (-1, 2]$ and $B = (1, 3)$ sketch the cartesian product $A \times B$



Venn diagrams 维恩图 ☆.

如 Compute $D = A-(B-C)$ A, B, C is any three sets.



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三. Graph Theory 图论

~~vetricle~~ vertices 顶点 edges 边 path 路径.

$V(G)$ 为 G 中点集. $|V|$ 顶点数表示 G 的 degree.

$E(G)$ 为 G 中边集. $|E|$ 边数表示 G 的 size

某个顶点的度数为该点的邻点数 adjacent. 两点 adjacent: 有边相连

度数为奇数的点叫做奇点. 度数为偶数的点叫做偶点.

complete graph

Every distinct pair of vertices is connected by an edge, need edges C_n^2

Isomorphic 同构

① 存在 $f: V(F) \to V(G)$ 一一双射. (象唯一, 原象唯一)

② All adjecenies are preserved

Connected

任意两点有路径相连 (多条边构成)

Cycle

存在一条封闭路径起点与终点一致.

$G \xrightarrow{\text{no cycles}} \text{forest} \xrightarrow{\text{connected}} \text{tree}$

Subgraph 子图

① $V(H) \subseteq V(G)$

② $E(H) \subseteq E(G)$

③ H 中出现的任一对连接点在 G 中也相连

△ 若 $V(H) = V(G)$ spaning subgraph of G.

if H is a tree, H is a spanning tree of G.

No.
Date.

四. Algorithms 算法学 (看程序写答案)

性质: ① process can't be ambiguity
② 有限步数后一定会有solution
③ The process is deterministic
(Using the same input value will always produce the same solution)

五. Combinatorics 组合学

$P(n,k) = n(n-1)(n-2)\cdots[n-(k-1)]$ ↙order makes differences.

$C(n,k) = \frac{n!}{k!(n-k)!} = \binom{n}{k}$ is called binomial cofficient 二项式系数.

二项式定理

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

With Repetitions Allowed 可放回.

如 10个球 编号 1-10. 取3次. 每次取了放回去, 三位数有多少个.

$P(n,k)$ with repetitions. n^k

$C(n,k)$ with repetitions C_{n+k-1}^{k}

抽屉原理 The pigeonhole principle

disjoint?

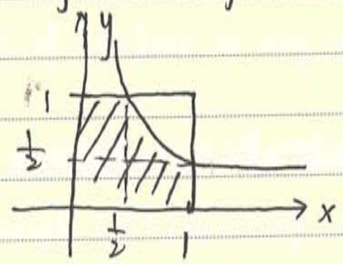
No.
Date.

六 Probability and statistics 概率论与统计.

face card 指扑克牌中的 J.Q.K.

eg: Two points. x and y. are selected at random in the interval [0,1]. what's the probability that the product xy will be less than $\frac{1}{2}$?

$xy < \frac{1}{2}$ $y < \frac{1}{2}$.



$$S = 1 - \int_{\frac{1}{2}}^{1} \left(1 - \frac{1}{2x}\right) dx$$

$$= 1 - \left(x - \frac{1}{2}\ln x\right)\Big|_{\frac{1}{2}}^{1}$$

$$= 1 - \left(1 - \frac{1}{2}\cdot 0\right) + \left(\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\ln 2$$

probability spaces 概率空间.

A Boolean algebra set on S 一定包括全集 空集.

$P(\phi) = 0$ $P(S) = 1$ $P(A \cup B) = P(A) + P(B)$.

independent events 独立事件 $P(AB) = P(A) \cdot P(B)$

exclusive events 互斥事件 $P(A+B) = P(A) + P(B)$.

$P(A+B) = P(A) + P(B) - P(AB)$.

Bernoulli trials 伯努利实验

$C_n^k p^k (1-p)^{n-k}$

random variables 随机变量

$P(t_1 < x \le t_2) = F_x(t_2) - F_x(t_1)$

$F_x(t) = P(X \le t)$ 有等号.

$P(t_1 \le X \le t_2) = F_x(t_2) - F_x(t_1) + P(X = t_1)$

$F_x(t)$ is called the distribution function of X. 分布函数.

$f_x(t)$ is called the probability density function of X 概率密度函数.

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期望 $E(x)=\int_{-\infty}^{+\infty} x f(x)\,dx$

方差 $Var(x)=\sigma^2(x) = \int_{-\infty}^{+\infty}[x-E(x)]^2 f(x)\,dx$

标准差 standard deviation $\sigma(x)=\sqrt{Var(x)}$

The normal distribution 正态分布.

$$f_X(t)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-t)^2}{2\sigma^2}}$$

标准正态分布 standard normal probability density.

$$f_Z(u)=\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$$

重要积分. $\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx=\frac{1}{2}$

二项分布的大数近似

$$P(a_1\le X\le a_2)=\sum_{k=a_1}^{a_2} C_n^k p^k(1-p)^{n-k} = \phi\left(\frac{a_2-\mu+\frac{1}{2}}{\sigma}\right)-\phi\left(\frac{a_1-\mu+\frac{1}{2}}{\sigma}\right)$$

$\mu=np$

$\sigma=\sqrt{np(1-p)}$.

$$\int_0^{+\infty}\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{2}}\,d\frac{x}{\sqrt{2}}=\frac{1}{2}$$

$$\int_0^{+\infty}\frac{1}{\sqrt{\pi}}e^{-x^2}\,dx=\frac{1}{2}$$

$$\int_0^{+\infty}e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$$

七. Point-set Topology 拓扑学.

X is an unempty set. A topology on X, denoted by T, is a family of subsets of X and satisfy the following:

1. ϕ and X are in T.
2. If ~~O~~ O_1, O_2 are in T, so their intersection, $O_1 \cap O_2$.
3. If $\{O_i\}_{i \in I}$ is any collection of sets from T, then their union, is also in T

X的拓扑是X所有子集的集合

A couple (X,T) consisting of a set X and a topology T in X is called a topological space.

Neighborhood 邻域.

(X,T) and $x \in X$. Any open set $U \in T$, containing x is called a neighborhood of x. ~~Any open set~~ Write as "nbd $U(x)$".

✓ Let x be a set and $T = \{\phi, X\}$, the topology T is called the indiscrete topology.
凝聚拓扑

✓ let X be a set and $D = P(X)$ be the family of all subsets of X. D is called the discrete topology.
离散拓扑

离散
~~连续~~拓扑中每个集合都为开集

eg: $X = \{0,1\}$. the discrete topology is $D = \{\phi, \{0\}, \{1\}, \{0,1\}\}$

Each topology is a subset of P(X). power of X

X的子拓扑交集仍为X拓扑. intersection

$$\bigcap_{\omega} T_\omega = \{U \mid \forall \omega \in \Omega, U \in T_\omega\}.$$

X的并集则不一定 union.

拓扑学中 $A^c = X - A$ $\bar{A}$ 为 closure

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A为开集条件：所有点均为内部点

eg：real line R 是开集

Accumulation Point

a为 accumulation point (limit point) of A：所有的开集U，that 至少 contains A中的一个点 different from A，and is denoted by A'

eg：$A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\}$ point $a=0$ is the accumulation point
$A' = 0$.

$Q' = R$.

闭集 Closed set

A为闭集条件：A^c 为开集 $(A^c = X - A)$

△ X和ϕ 既是开集又是闭集

无限 (infinite) 闭集的交集 (intersection) 是闭集，并集 (union) 不一定

$]a,b[= (a,b)$

$]a,b] = (a,b]$

Closure 将开集A补全的最小闭集

eg：$A = (0,1]$ $\bar{A} = [0,1]$

$B = \{\frac{1}{n} \mid n = 1, 2, 3 \cdots\}$ $\bar{B} = B \cup \{0\}$.

对于任意集合 A. $A \subset \bar{A}$. A为闭集 $\Leftrightarrow A = \bar{A}$

四条性质

(1) $A \subset B \Rightarrow \bar{A} \subset \bar{B}$

(2) $\bar{\bar{A}} = \bar{A}$

(3) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ ✓ $\overline{A \cap B} = \bar{A} \cap \bar{B}$ × 实际上 $\overline{\bigcap_w A_w} \subset \bigcap_w \bar{A}_w$

(4) $\bar{\phi} = \phi$

Interior A中的最大开集

Closeur 含A的最小闭集

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① $\overline{A} = A' \cup A$

② A为闭集 $\Longleftrightarrow A' \subset A$

Interior 内集.

The $Int(A)$ of A is the largest open set contained in A.

eg: $A = (0,1]$ Then $Int(A) = (0,1)$

$A = \{\frac{1}{n} \mid n = 1,2,3,\dots\}$ Then $Int(A) = \phi$

✓ A是开集 $\Longleftrightarrow A = Int(A)$

✓ $Int(A) = (\overline{A^c})^c$

边境 Boundary

$Fr(A)$ is a closed set

$Fr(A) = \overline{A} \cap \overline{A^c}$

性质

① $\overline{A} = Int(A) \cup Fr(A)$

② $Int(A) \cap Fr(A) = \phi$

③ $Fr(A) = \overline{A} - Int(A)$

Dense set

A set $D \subset X$ is said to be dense in X if

$\overline{D} = X$

✓ X is dense in X, and X is the only closed set dense in X.

✓ $\mathbb{Q}$ is dense in $\mathbb{R}$.

~~Neighborhood~~ System

~~Let x be a point in a topology space X, a subset N of X is a neighborhood of x~~

~~if N conta~~

Neighborhood System

The class of all neighborhoods of $x \in X$, denoted by N_x, is called the neighborhood system of x. X的子集中所有包含 x 的集合

性质

① $N_x \neq \phi$

② x belongs to each member of N_x

③ N_x 中两个集合的交集也属于 N_x.

④. 任何包含 N_x member 的都属于 N_x

子空间的子空间是全空间的子空间.

传递性: 若 Y为X子空间. 如果 $A \subset Y$ is $\overset{\text{close}}{\text{open}}$ in Y and Y is $\overset{\text{close}}{\text{open}}$ in X. then A is $\overset{\text{close}}{\text{open}}$ in X.

基 Basis.

(X,T) 是一个拓扑空间. A family $B \subset T$ is called a basis for T if 所有T的成员开集, 都是B成员的并集.

Let D be the discrete topology on X. Then

$$B = \{\{x\} \mid x \in X\}$$

is a basis for D.

$B = \{(a,b) \mid$ ~~a,b~~ $\in R$ and $a < b\}$ is a basis of R. 因为R中任何开区间都可用B中union表示

R^n has a countable basis.

基地 Base

B is a family of subsets of (X,T). Family B is a base for some topology on X if and only if it has the following two properties:

A) $X = \cup\{A \mid A \in B\}$.

B) 对于任意的 $A_1, A_2 \in B$. $A_1 \cap A_2$ is B的成员所组成的并集.

Basis / Base / Subbase.

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Date.

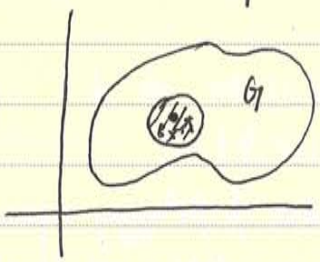
子基地

$S \subset T$, is a subbase for the topology T on X if 有限的S中成员的交集可以组成T的一个基地

△ The class of all infinite open intervals is a subbase for R^1.

Local Base.

Consider the Euclidean topology in R^2. Let $x \in R^2$. Then the family of all open balls with centers at x is a local base at x. Indeed, any open set G containing x also contains an open ball B_x with center at x.



B_x is ~~the~~ a local base at x.

Converge ?? R-255.

Continuous functions 连续函数.

$f: P(X) \to P(Y)$ $\quad\quad$ $f^{-1}: P(Y) \to P(X)$

f^{-1}满足: ① $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

② $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

③ $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$

A function $f: X \to Y$ is called continuous if the inverse image of each open set in Y is open in X.

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以下几点等价
① f连续
② Y中闭集的原像在X中也为闭集
③ Y的基的每个member的原象在X中都为开集
④ 对于every $A \subset X$
$f(\bar{A}) \subset \overline{f(A)}$ ☆
⑤ 对于 every $B \subset Y$
$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ ☆
⑥ $V(x)$ is a neighborhood of x in X
$U(f(x))$ is a neighborhood of $f(x)$ in Y.

Sequential Continuity at a Point
A function $f: X \to Y$ is sequentially continuous at $a \in X$ if for every sequence a_n in X converging to a, the sequence $(f(a_n))$ converges to $f(a)$;
that is:
$$(a_n \to a) \Rightarrow f(a_n) \to f(a))$$

如果f是一个连续函数,则开集的像是开集,闭集的像是闭集.

Homeomorphism
双射.
A continuous bijective function $f: X \to Y$, such that $f^{-1}: Y \to X$ is also continuous, is called a homeomorphism and is denoted by
$f: X \cong Y$.

拓扑性质?

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Date.

int(A)　A中最大开集

ext(A)　与A无交集的最大开集

cl(A)　含A的最小闭集 $= int(A) \cup bd(A)$

bd(A)　将A补全为闭集的元素集合.

A'　accumulate point　(就是cl(A)?)

Finer and coarser

If T_1 and T_2 are topologies on X, then we say T_1 is finer than T_2 if $T_1 \supseteq T_2$. and T_2 is coarser than T_1. Thus the indiscrete topology is the coarsest and the discrete topology is the finest.

eg: X be the set $\{a,b,c,d\}$.

$T_1 = \{\phi, \{a,b\}, \{c,d\}, X\}$ is a topology on X.

$A = \{\phi, \{a\}, \{a,b\}, \{c,d\}, X\}$ is not a topology on X. for the union $\{a\}$ and $\{c,d\}$ are not in A

so $A' = \{\phi, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}, X\}$ is a topology on X

eg: $(0,1) \cup (1,2)$

$int(A) = A$　$ext(A) = (-\infty, 0) \cup (2, +\infty)$ 一定为开集. 所以不含!

$bd(A) = \{0,1,2\}$　$cl(A) = [0,2]$　$A' = [0,2]$

开集可以用唯一的可数个开集的并集表示.

相交相连. 连通 Connectedness

如果存在不相交的非空开集 O_1, O_2 使得 $O_1 \cup O_2 = X$. 那么X是不连通的. 相反, 如果T中没有一对并集为X的不相交子集, 则X连通.

① 两连通区域的交集, 并集连通

② 两个连通空间的 carbesian 积也是连通的.

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Compaitness 紧性

A covering of X is a collection of subsets of X whose union is X.
If every open covering of X contains a finite subcollection that also covers X, then X is said to be a compact space.

(0,1) is not compact, [0,1] is compact

① X is a compact topological space, and let S be a subset of X. If S is closed, then it's compact.

② The cartesian product of compact spaces is compact.

定义 norm of a point $x=(x_1, x_2 \cdots x_n)$ in R^n as the real number

$$\|X\| = \sqrt{(x_1)^2+(x_2)^2+\cdots+(x_n)^2}$$

A subset A of R^n is said to be bounded if 存在正常数M，对任意点 x in A，$\|x\|$小于M.

subset of R^n 紧密的条件：closed, bounded.

$d(x,y)$ is the distance between x and y. The function d is said to be a metric on X.

$B_d(x,\varepsilon)=\{x' \in X : d(x,x')<\varepsilon\}$ is called an ε-ball. The (open) ball of radius ε centered on x.

函数的连续性 depends on the topologies defined on the domain and range space.

$f: (X_1, T_1) \to (X_2, T_2)$

The set $f^{-1}(C)$ is closed in X_1 for every closed subset C of X_2 if and only if the map $(X_1,T_1) \to (X_2,T_2)$ is continuous.

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八. Real analysis 实数分析

☆ Lebesgue integration

最小上界 $\text{lub}\,x \equiv \sup x$

最大下界 $\text{glb}\,x \equiv \inf x$.

if real is placed by rational, statements may be false.

柯西序列.

$(x_n)_{n=1}^{\infty}$ is called a Cauchy sequence if, for every $\varepsilon > 0$ (no matter how small), there exists an integer N such that for every pair of $m, n > N$. $|x_m - x_n| < \varepsilon$.

△ Every Cauchy sequence of real numbers converges.

$\mathbb{R}$ is complete — it has no holes!

勒贝格测度 Lebesgue Measurable Functions.

$\mu^*(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i), \text{ for } A \subseteq \bigcup_{i=1}^{\infty}[a_i, b_i)\}$

we allow $\mu^*(A) = \infty$.

A measurable set M for which $\mu(M) = 0$ is said to be a set of measure zero.

$\mathbb{R}$中所有开集、闭集均可测. $\mathbb{R}$中任何有限或可数无限的子集也是勒贝格可测的.

可测集的 complement 补集也可测.

① 若 $M = \{m\}$ is a one-element subset of $\mathbb{R}$. then $\mu(M) = 0$.

② 若 M is a finite or countably infinite subset of $\mathbb{R}$, then $\mu(M) = 0$.

$\mu(\mathbb{Z}) = \mu(\mathbb{Q}) = 0$.

eg: 若 $M = (-1, 2] \cup (6, 8]$ then $\mu(M) = 5 = 2 - (-1) + 8 - 6 = 5$

如果 M 有无穷个区间并且可测, 则 $\mu(M) = \infty$.

③ 若 M_1 and M_2 are elements of $\mathcal{M}$ and $M_1 \subseteq M_2$. then $\mu(M_1) \leq \mu(M_2)$

如果 A 是区间 $[a, b]$ 和 $[c, d]$ 的笛卡尔积, 则它是一个长方形 $\mu(A) = (b-a)(c-d)$

一维: 长度 二维: 面积 三维: 体积

勒贝格可测函数

包含 E 的下确界的测度叫作 E 的 L 外测度 $m^*(E)$

— — 上确界 — —— 内 、、 $m_*(E)$

若 $m^*(E) = m_*(E)$ 则称 E 为勒贝格可测集

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△无界点集的测度可能为有限值,可能为无穷大

$\mu(\mathbb{Q})=0$

$E=(0,+\infty)\quad \mu(E)=\infty$

设 $E\subset R$ 为任一可测集(有界或无界). $f(x)$ 为定义在 E 上的实值函数.

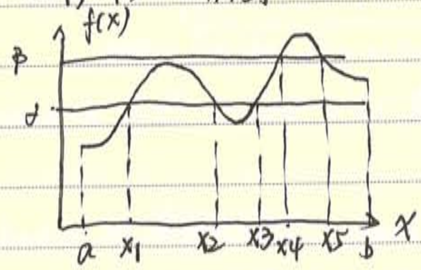
若 $\forall a\in R$. E 的子集

$E(f\geq a)=\{x\mid f(x)\geq a, x\in E\}$.

都为有限可测集. 则 $f(x)$ 为 E 上的可测函数

$E(f\geq a)=[x_1,x_2]\cup[x_3,b]$

$E(f>\beta)=[x_4,x_5]$



① 定义在 R 上连续函数都是 L 可测函数.

② 定义在零测集 E 上的任何函数 $f(x)$ 都是 L 可测函数

特征函数

$$\chi_A(x)=\begin{cases}0 & \text{if } x\notin A\\ 1 & \text{if } x\in A\end{cases}$$

eg $\quad s=2\chi_{(-3,-2)}-3\chi_{[-1,2)}+4\chi_{[4,6]}$

勒贝格积分

函数 $s=\sum_{i=1}^{n}a_i\chi_{A_i}$

则 $\int s\,d\mu=\sum_{i=1}^{n}a_i\mu(A_i)$

eg: for $\quad s=2\chi_{(-3,-2)}-3\chi_{(-1,2)}+4\chi_{[4,6]}$

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$\int s\,d\mu = 2\times 1 - 3\times 3 + 4\times 2 = 1.$

$\int f\,d\mu = \sup\{\int s\,d\mu\}.$

$f = f^+ - f^- = \frac{1}{2}(|f|+f) - \frac{1}{2}(|f|-f).$

$\therefore \int f\,d\mu = \int f^+ d\mu - \int f^- d\mu.$

勒贝格积分与黎曼积分.

approaches for discontinuous.

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九. Complex variable 复变函数

1. 复数运算

$+, -, \times, \div$

eg: $(2-3i)(-5+i)$

$\frac{2-3i}{-5+i}$

2. 极坐标

$z = re^{i\varphi}$ $\quad z^2 = z \cdot \bar{z}$

$(-\pi < \varphi \le \pi)$

欧拉公式 $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

3. 复根.

$z^n = 1 \quad = (re^{i\varphi})^n = 1 \cdot e^{i2\pi k} = r^n e^{in\varphi}$

$\because 1 = 1 \cdot e^{i(2\pi k)} \quad \therefore r = 1 \quad \varphi = \frac{2\pi k}{n} \quad (k = 0, 1, 2, \cdots n-1)$

eg: What are the fourth roots of $16i$

$z^4 = 16i = 16e^{i(\frac{\pi}{2} + 2k\pi)} = |z|^4 e^{i4\cdot\varphi}$

$\therefore |z| = 2 \quad 4\varphi = \frac{\pi}{2} + 2k\pi \quad \varphi = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

where $z = 2e^{i\frac{\pi}{8}}$ is called the principle fourth root.

$\log z = \log r + i(\theta + 2\pi k)$

We use the unique $\mathrm{Log}\, z$ to represent the principal logarithm of z

$\mathrm{Log}\, z = \log|z| + i\mathrm{Arg}\, z$

eg: The principle logarithm of $-e^2 i = \log|e^2 i| + i(-\frac{\pi}{2}) = 2 - i\cdot\frac{\pi}{2}$

principle value

The principal value of z^w is equal to $e^{w\operatorname{Log} z}$, where $\operatorname{Log} z$ denotes the principal value of $\log z$. (or say the principal logarithms of z)

eg: What's the principle value of i^i?

$$z^w = e^{w\log z} = e^{i\operatorname{Log} i} = e^{(0+i\cdot\frac{\pi}{2})i} = e^{-\frac{\pi}{2}}$$

$\log 1 = 0$ ☆

☆ $e^{\pi i} + 1 = 0$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

eg: compute z. $\sin z = 2$.

$$\frac{e^{iz}-e^{-iz}}{2i} = 2 \qquad \therefore e^{iz}-e^{-iz} = 4i$$

~~$\cos z + i\sin z - \cos z$~~ $\quad e^{i(x+iy)} - e^{-i(x+iy)} = 4i$

$\therefore e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x) = 4i$

$\cos x = 0 \quad \sin x = 1 \quad e^{y} + e^{-y} = 4$

$x = \frac{\pi}{2} + 2k\pi$ $\quad$ ~~e^{y}~~ $\quad y = \log_e(2\pm\sqrt{3})$

$\therefore z = \left(\frac{\pi}{2} + 2k\pi\right) + i\ln(2\pm\sqrt{3})$

The hyperbolic functions

$$\cosh x = \frac{e^{x}+e^{-x}}{2} \qquad \sinh x = \frac{e^{x}-e^{-x}}{2}$$

① $\cosh(iz) = \cos z$ $\qquad$ ② $\sinh(iz) = i\sin z$

$\cos(iz) = \cosh z$ $\qquad$ $\sin(iz) = i\sinh z$

eg: $\cos z = \cos(x+yi) = \cos x\cos(yi) - \sin x\sin(yi) = \cos x\cosh y - i\sin x\sinh y$

若$f(z)$在$z=z_0$可导则极限 $\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$存在(充要条件)

eg: If $f(z) = z - \arctan z$. solve the equation $f'(z) = \frac{4}{3}$

$f'(z) = 1 - \frac{1}{1+z^2}$ $\qquad \therefore 1 - \frac{1}{(1+z^2)} = \frac{4}{3}$

$-\frac{1}{3} = \frac{1}{(1+z^2)} \Rightarrow$ ~~$(1+z)^2 = -3$~~ ~~$1+z = \pm\sqrt{3}i$~~

$\to 1+z^2 = -3$

$z^2 = -4 \Rightarrow z = \pm 2i$

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The Cauchy-Riemann Equation 柯西黎曼公式

若 $f(x,y)=u(x,y)+i(x,y)$ 可导

$\Leftrightarrow \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$.

若函数解析，则其实部、虚部均满足拉普拉斯方程

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \end{cases}$$

对于复函数若其在 z_0 点解析，则它的微分 $f'(z)$ 在 z_0 点也解析

解析与可导的区别.

在某一点

在区域内所有点可导才称作解析.

Complex Line Integrals 复函数积分

eg: 求 $\int f(z)dz \quad f(z)=z^2-iz$ 路径为 $z(t)=x(t)+iy(t)$

$$\begin{cases} x=t \\ y=2t^2 \end{cases} \quad \text{from } 0\xrightarrow{t}1$$

法一 $z=t+i2t^2$

$\therefore \int f(z)dz=\int_0^1 (t+i2t^2)^2-i(t+i2t^2)\, d(t+i2t^2)$

$=-\frac{5}{3}+\frac{5}{6}i$

法二 $\int f(z)dz=\int z^2-iz\, dz=\left(\frac{1}{3}z^3-\frac{1}{2}iz^2\right)\Big|_0^{1+2i}=-\frac{5}{3}+\frac{5}{6}i$

柯西定理

若$f(z)$在区域D解析，则D中任一闭曲线上积分

$$\oint_C f(z)dz=0.$$

莫雷拉定理　　(柯西定理逆定理)

若$f(z)$在D内连续，且沿D内任一可求长闭曲线γ的积分

$$\oint_\gamma f(z)dz=0$$

则称$f(z)$在D内解析.

☆ 柯西积分公式.

如果$f(z)$在绕z_0的一条封闭路径及其所围区域内解析，则

$$f(z_0)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-z_0}dz$$

高阶　$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^{n+1}}dz$$

柯西微分估计

$f(z)$ is analytic on and within $|z-z_0|=r$. the circle of radius r centered at z_0. If M denotes the maximum value of $|f(z)|$ on the circle, then:

$$|f^{(n)}(z_0)|\le\frac{n!M}{r^n}$$

泰勒展开

常见的函数泰勒展开式

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}\quad(|z|<\infty)$$

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\quad(|z|<1)$$

$$\frac{1}{1+z}=\sum_{n=0}^{\infty}(-1)^n z^n\quad(|z|<1)$$

$$\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\quad(|z|<\infty)$$

$$\cos z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}\quad(|z|<\infty)$$

$$\ln(1+z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{n+1}}{n+1}\quad(|z|<1)$$

$$(1+z)^\alpha=1+\alpha z+\frac{\alpha(\alpha-1)}{2!}z^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}z^n\quad(|z|<1)$$

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例. 将函数 $f(z)=\frac{1}{(1-z)^2}$ 在 $z=i$ 展为泰勒级数

$f(z)=\left(\frac{1}{(1-z)}\right)'$

$g(z)=\frac{1}{1-z}=\frac{1}{1-i-(z-i)}=\frac{1}{1-i}\cdot\frac{1}{1-\frac{z-i}{1+i}}$

$=\frac{1}{1+i}\cdot\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}}$ $\left|\frac{z-i}{1+i}\right|<1 \Rightarrow |z-i|<2$

$\therefore f(z)=g'(z)\cancel{=\frac{1}{(1+i)^{n+1}}\cdot\sum_{n=0}^{\infty}}=\sum_{n=1}^{\infty}\frac{n(z-i)^{n-1}}{(1-i)^{n+1}}$

$=\sum_{n=0}^{\infty}\frac{n+1}{(1-i)^{n+2}}(z-i)^n$

将函数 $f(z)=e^z\sin z$ 在 $z=0$ 展开为幂级数

$f(z)=e^z\cdot\frac{e^{iz}-e^{-iz}}{2i}=\frac{1}{2i}\left[e^{z(1+i)}-e^{z(1-i)}\right]$

$=\frac{1}{2i}\left(e^{1+i}\cdot\sum_{n=0}^{\infty}\frac{z^n}{n!}\right.$ ✗ $e^{1+i}\cdot e^z=e^{1+i+z}$ ✗

$=\frac{1}{2i}\left(\sum_{n=0}^{\infty}\frac{(1+i)^n z^n}{n!}-\sum_{n=0}^{\infty}\frac{(1-i)^n z^n}{n!}\right)$

洛朗级数

有奇点的幂级数

$\underbrace{\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}}_{\text{主部分}}+\underbrace{\sum_{n=0}^{\infty}a_n(z-z_0)^n}_{\text{解析部分 (用泰勒)}}$

不在~~解析~~收敛区域内的用倒数变进去

eg: 展开 $f(z)=\frac{4z}{(z-1)(z-3)^2}$

(1) $1<|z|<3$

(2) $0<|z-3|<2$

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解：$\frac{4z}{(z-1)(z-3)^2} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{Cx+D}{(z-3)^2}$

$= \frac{1}{z-1} + \frac{-1}{z-3} + \frac{6}{(z-3)^2}$

$= -\frac{1}{1-z} + \frac{1}{3-z} + \frac{6}{(z-3)^2}$

$= -\frac{1}{z}\cdot\frac{1}{\frac{1}{z}-1} + \frac{1}{3-z} + \frac{6}{(z-3)^2}$

$= \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} + \frac{1}{3-z} + \frac{6}{(z-3)^2}$

$\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$

留数定理

$a_{-1}\cdot 2\pi i = \oint_C f(z)\,dz$

其中 $a_{-1} = \mathrm{Res}(z_0, f)$，称为留数

设闭曲线 C 中包含一个极点 z_0，阶数为 k

则 $a_{-1} = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k\cdot f(z)\right]$，$2\pi i\, a_{-1} = \oint_C f(z)\,dz$

若包含多个极点

$\oint_C f(z)\,dz = 2\pi i \sum \mathrm{Res}(z_i, f)$.

等价
- 原命题 $a \to b$
- 逆命题 $b \to a$
- 否命题 $\bar{a} \to \bar{b}$
- 逆否命题 $\bar{b} \to \bar{a}$

（逆命题与否命题）等价

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Numerical Analysis

二分法 bisection method

$$c = a + \frac{b-a}{2}$$

Newton's method 牛顿法

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

△ The initial x_0 is very important

Attention

1. disjoint: $A \cap B = \phi$
2. symmetric difference: $A \triangle B = (A-B) \cup (B-A)$
3. cartesian product (笛卡尔积) $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

   一维 × 一维 → 二维. 长方形

4. card elements in the set

   eg: $n = \{1, 2, \cdots n\}$ card $= n$ $P(n) = 2^n$

   countable infinite card $Z^+ = \aleph_0$

   uncountable infinite card $R = 2^{\aleph_0} = c$

5. degree $|V|$ size $|E|$
6. isomorphic 所有邻点被保留(结构未被破坏)
7. trace 行迹 路径中边不同 → cruckit

   track 轨迹 路径中顶点不同 → cycle.

graph $\xrightarrow{\text{no cycle}}$ forest $\xrightarrow{\text{connected}}$ tree

8. matric incidence matric

边

	e_1	e_2	e_3	e_4	e_5
v_1	1				
v_2	0				
v_3	1				
v_4	0				

顶点

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9. independent $P(AB)=P(A)P(B)$
exclusive $P(A+B)=P(A)+P(B)$
$P(A+B)=P(A)+P(B)-P(AB)$

10. distrution function $F_X(t)=P\{X\le t\}$.
probability density function $f_X(t)$
expectation $E(X)=\mu(X)=\int_{-\infty}^{\infty} t f_X(t)\,dt$
variance $Var(X)=\sigma^2(X)=\int_{-\infty}^{+\infty}[t-\mu(X)]^2 f_X(t)\,dt$
standard ~~vari~~ deviation. $\sigma(X)=\sqrt{\sigma^2(X)}$

11. normal distribution
$f_X(t)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(t-\mu)^2/2\sigma^2}$
standard normal probability density.
$f_Z(u)=\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$

12. n is large.
$P(a_1\le X\le a_2)=\sum_{k=a_1}^{a_2} C_n^k p^k q^{n-k} = \phi\left(\frac{a_2-\mu+\frac{1}{2}}{\sigma}\right)-\phi\left(\frac{a_1-\mu+\frac{1}{2}}{\sigma}\right)$
$\mu=np$ $\sigma=\sqrt{npq}$

13. indiscrete topology $T=\{\phi, X\}$
discrete topology $T=\{\phi,\{a,b\},\{c,d\},X\}$. (not discrete)
$\downarrow$
P(T)
遇到 P(T). card 2^n

14. interior int(A) 开集
exterior ext(A) 开集
boundary bd(A) 元素集
closure cl(A) 闭集 (limit point, seems to equal cl(A)?)

15. product topology ~~[1×2]~~ $(1,2]\times(1,2)$ 既不开,也不闭

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16. connectedness

(X, T) be a topological space. If there exist disjoint, nonempty open set, O_1 and O_2, such that $O_1 \cup O_2 = X$. then X is said to be disconnected.

17. compactness

If every open covering of X contains a finite subcollection that also covers X, then X is said to be a compact space.

判断方法

① norm $\|x\| = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2} < M$

② closed

18. Metric space 距离空间.

$d(x, x') = \|x - x'\| = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + \cdots + (x_n - x_n')^2}$

19. continuous function

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ then it is continuous at x_0.

连续性依赖于拓扑的定义域和 range space.

20. μ for real analysis

$\mu(\mathbb{Z}) = \mu(\mathbb{Q}) = 0$

$\int s \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$

tautology: for every possible value, it is true

eg: $\neg(\neg A) \leftrightarrow A$

Precalculus

1. domain 定义域
2. compose functions.

eg: $f: A \to B$ $\quad g: B \to C$

the composite $(g \circ f)(x) = g(f(x))$

3. inverse functions 反函数.

one-to-one $\quad x_1 \neq x_2 \to f(x_1) \neq f(x_2)$ $\quad$ eg: x^3 is one-to-one. x^2 is not injective

onto $\quad f: A \to B$ $\quad$ B中任何element在A中都有原象 $\quad$ bijective

如定义在 $R \to R$ 的 $g(x) = x^2$ $\quad g(x) = -4$ 在R上，但无原象

如果满足双射. 则 $f(x) = y$ 有反函数 $f^{-1}(y) = x$.

Analytic geometry

1. Line

$y - y_1 = m(x - x_1)$

$ax + by + c = 0$.

2. Parabolas 抛物线

$y = \pm\frac{1}{4p}x^2$ $\quad$ 焦点 focus $\quad y = \cancel{(p,0)}\ (0,p) \quad \cancel{(-p,0)}\ (0,-p)$ $\quad$ directrix 准线 $y = \pm p$.

$x = \pm\frac{1}{4p}y^2$ $\quad\quad x = (p,0) \quad (-p,0)$

3. circle 圆

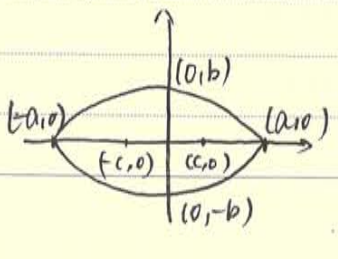
$x^2 + y^2 = a^2$

$(x-h)^2 + (y-k)^2 = a^2$ $\quad$ 切线 $k_1 k_2 = -1$

4. Ellipse 椭圆.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$ $\quad$ 长轴 major axis $\quad$ 短轴 short axis

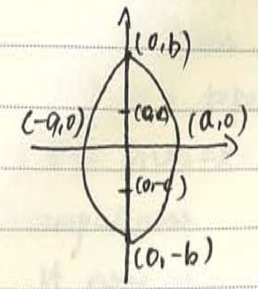
焦点 $c = \pm\sqrt{a^2 - b^2}$ $\quad$ 离心率 eccentricity. $\quad$ 长轴顶点叫 verticle



$e = \frac{c}{a}$

major axis length = $2a$

minor axis length = $2b$.



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (b>a)$

$e=\frac{c}{b}$ 长轴   准线 directrix $y=\pm\frac{b^2}{c}$ 长轴

major axis length $2b$

minor axis length $2a$

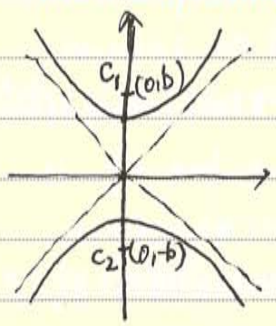
椭圆上每点到两焦点距离之和为长轴长度 $2a$ / $2b$.

5. Hyperbolas 双曲线

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ 渐近线 asymptotes

$c=\sqrt{a^2+b^2}$ 焦点 foci $y=\pm\frac{b}{a}x$.

双曲线上每点到焦点距离之差为 $2a$ / $2b$



$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

$||AC_1|-|AC_2||=2b$.

$\cancel{e=\frac{b}{a}}$ $e=\frac{c}{b}>1$ 实轴长

准线 $y=\pm\frac{a^2}{c}$

椭圆平面内到定点 $F(c,0)$ 的距离和到 $x=\frac{a^2}{c}$ 距离之比为离心率

椭圆离心率小于1. 双曲线离心率大于1

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Polynomial equation 多项式方程

$p(x) = q(x)d(x) + r(x)$

~~quotient~~ quotient remainder.

1. the rational roots theorem 有理根定理

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

若 $p(x)=0$ 有有理根. $x_i = \frac{p}{q}$ $(p,q)=1$ 则 $p \mid a_0$, $q \mid a_n$.

2. the conjugate radical roots theorem

若 $p(x)=0$ 有根 $x = s + t\sqrt{u}$, 则必有根 $s - t\sqrt{u}$.

3. the complex conjugate roots theorem

若 $p(x)=0$ 有根 $x = s + ti$, 则必有根 $s - ti$

4. sum and product of the roots

$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ $(a_n \neq 0)$

sum of the roots. $= -\frac{a_{n-1}}{a_n}$

product of the roots $= (-1)^n \frac{a_0}{a_n}$

特别地 $ax^2 + bx + c = 0$

$$\begin{cases} x_1 + x_2 = -\frac{b}{a} \\ x_1 x_2 = \frac{c}{a} \end{cases}$$

eg: Find the positive rational roots of the equation $3x^4 - 7x^3 + 5x^2 - 7x + 2 = 0$.

由有理根定理 x 可能为 $\frac{2}{3}, \frac{1}{3}, 2, 1$

代入得 $x_1 = \frac{1}{3}$, $x_2 = 2$

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Logarithms 对数

$y = \log_b x \iff b^y = x$

$x > 0$, 若 $b > 1$ y is increasing as x increases.

若 $0 < b < 1$ y is decreasing

① $\log_b(x_1 x_2) = \log_b x_1 + \log_b x_2$

② $\log_b(x^a) = a\log_b x$

③ $b^{\log_b x} = x$

④ 换底公式 change-of-base formula

$\log_a x = \dfrac{\log_b x}{\log_b a}$

$e = 2.718$ natural logarithm function

eg: Solve for x: $4^x = 2^x + 3$

$(2^x)^2 - 2^x - 3 = 0$

$\Delta = 1 + 4 \times 3 = 13$

$2^x = \dfrac{1 \pm \sqrt{13}}{2}$

$\therefore x = \log_2 \dfrac{1 \pm \sqrt{13}}{2} = \log_2(1 \pm \sqrt{13}) - 1 = \log_2(1 + \sqrt{13}) - 1$

eg: Simplify $[\log_{xy}(x^y)][1 + \log_x y]$

$\log_{xy}(x^y) \cdot \log_x xy = y\log_{xy} x \cdot \log_x xy = y$

$\cancel{\log_a x = \dfrac{\log_b x}{\log_b a}}$

$\cancel{\log_{xy} xy = \dfrac{\log_x xy}{\quad}}$

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Trigonometry

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$

$1+\tan^2\theta = \sec^2\theta$ $1+\cot^2\theta = \csc^2\theta$ $\sin^2\theta + \cos^2\theta = 1$

$\sin(\alpha+\beta)$ $\sin(\alpha-\beta)$ $\cos(\alpha+\beta)$ $\cos(\alpha-\beta)$

$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ $\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$.

$\sin2\theta = 2\sin\theta\cos\theta$ $\cos2\theta = 2\cos^2\theta - 1$ $\tan2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

eg: Compute $\tan\frac{\pi}{12}$

$\tan\frac{\pi}{12} = \tan(\frac{\pi}{3}-\frac{\pi}{4})$

反三角函数.

~~Function~~ $\arcsin x$ $x\in[-1,1]$ $\arcsin x\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$\arccos x$ $x\in[-1,1]$ $\arccos x\in[0,\pi]$

$\arctan x$ $x\in R$ $\arctan x\in(-\frac{\pi}{2},\frac{\pi}{2})$

$\operatorname{arccsc} x$ $|x|\geq1$ $\operatorname{arccsc} x\in[-\frac{\pi}{2},\frac{\pi}{2}]\neq0$

eg: Evaluate $\arccos\frac{2}{\sqrt5}+\arccos\frac{3}{\sqrt{10}}$.

$\tan\alpha = \frac{1}{2}$ $\tan\beta = \frac{1}{3}$. $\alpha+\beta$

$\tan(\alpha+\beta) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$

$\therefore \alpha+\beta = 45^\circ$.

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Limits of sequences 数列极限

converge 收敛 diverge 发散

diverge to infinity 发散到正无穷

diverge to minus infinity 发散到负无穷

monotonic ~~单调~~ 单调

① $a_n \to A$ $\quad Ka_n \to KA$

② $a_n \to A$ $\quad b_n \to B$

$(a_n + b_n) \to A+B$

$(a_n - b_n) \to A-B$

$a_n b_n \to AB$

$\frac{a_n}{b_n} \to \frac{A}{B}$ $\quad (B \neq 0)$

③ K为正常数 $\frac{1}{n^K} \to 0$

$|K|>1$ $\quad (\frac{1}{K^n}) \to 0$

Sandwich theorem 夹逼定理

若 $a_n \to L$, $c_n \to L$ for every $n > N$ $\quad a_n \le b_n \le c_n$

则 $b_n \to L$

L'Hôpital's rule (连续.离散)

若 $a_n = f(n)$，则若 $f(x) \to L$, as $x \to \infty$, 那么 $a_n \to L$

Limits of function 函数极限

$\lim_{x \to a} f(x) = L$

左极限 $\lim_{x \to a^-} f(x)$ $\quad$ 右极限 $\lim_{x \to a^+} f(x)$

在 x_0 点存在极限的条件：左极限,右极限相等

性质：① $\lim\limits_{x \to a} x = a$ $\quad \lim\limits_{x \to a} K = K$ $\quad \lim\limits_{x \to a} x^n = a^n$

② $\lim\limits_{x \to a} f(x) = L_1$ and $\lim\limits_{x \to a} g(x) = L_2$.

$\lim_{x\to a}[f(x)+g(x)]=L_1+L_2$ $\lim_{x\to a}[f(x)-g(x)]=L_1-L_2$

$\lim_{x\to a}[f(x)g(x)]=L_1L_2$ $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{L_1}{L_2}$

夹逼定理.

$\lim_{x\to a}f(x)=L$ $\lim_{x\to a}h(x)=L$ 若 $f(x)\leq g(x)\leq h(x)$ 则 $\lim_{x\to a}g(x)=L$

Continuous Function 连续函数.

$f(x)$ 在 a 点, 连续 $\lim_{x\to a}f(x)=f(a)$

① 在 a 点有定义 ② 左右极限相等 且等于 a 点的值

The extreme value theorem 极值定理

$f(x)$ 是定义在闭区间 $[a,b]$ 上的连续函数. 则一定存在 $c,d\in[a,b]$ 使得 $f(c)\leq f(x)\leq f(d)$ 恒成立

Bolzano's theorem

$f(x)$ 是定义在闭区间 $[a,b]$ 上的连续函数. $f(a)\cdot f(b)$ 符号相反, 一定存在 $f(c)$, $c\in[a,b]$. $f(c)=0$

The Intermediate Value theorem 中值定理

$f(x)$ 是定义在闭区间 $[a,b]$ 上的连续函数. m 是 $f(x)$ 在 $[a,b]$ 最小值, M 是 $f(x)$ 在 $[a,b]$ 最大值, 则对于任意 Y. $m\leq Y\leq M$ 可以找到 $c\in[a,b]$. $f(c)=Y$.

Derivative 微分

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Leibniz notation 莱布尼茨表述

$$f'(x)=\frac{df}{dx}.$$

☆ ① $(f+g)'(x)=f'(x)+g'(x)$

② $(kf)'(x)=kf'(x)$

③ $(fg)'(x)=fg'(x)+f'(x)\cdot g(x)$

④ $\left(\frac{f}{g}\right)'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$

$f(x)$ 反函数在 (x_0,y_0) 的导数是 f 在 (x_0,y_0) 导数的倒数 $[f^{-1}(x)]'=\frac{1}{f'(x)}$

$$\frac{df}{dx}=\frac{df}{du}\cdot\frac{du}{dx}$$

几个重要微分

① $d(a^x)=\frac{1}{x\ln a}dx$ ② $d(\tan u)=\sec^2u\,du$ ③ $d(\cot u)=-\csc^2x\,du$

④ $d(\sec x)=\sec x\tan x\,dx$ ⑤ $d(\csc x)=-\csc x\cot x\,dx$

⑥ $d(\arctan x)=\frac{1}{1+x^2}$ ⑦ $d(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$

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Nomal line 法线

Linear approximation 线性近似

$f(b) = f'(a)(b-a) + f(a)$

eg: What's an approximate value of $e^{0.1}$.

$(e^x)' = e^x$ $e^0 = 1$

$\therefore e^{0.1} = 1 + e^0(0.1-0) = 1.1$

Implicit differentiation 隐函数求导

对等式两边用复合函数求导法则对两边求导.

eg: $x^2y^5 + y = 2$

$2xy^5 + x^2 \cdot 5y^4y' + y' = 0$

$\therefore y' = \dfrac{-2xy^5}{5x^2y^4+1}$

Higher-order derivation 高阶导数.

$f''(x)$ second derivation $f'''(x)$ third derivation

$f''(x) = \dfrac{d^2f(x)}{dx^2}$ $f'''(x) = \dfrac{d^3f(x)}{dx^3}$

Curve sketching 曲线描绘

① $f'(x)=0$ stationary point / critical point 极点, 驻点,

$f'(x)$ 在 x_0 点不存在 critical point 极点,

② $f''(x)=0$ inflection point 拐点,

③ 若 $f'(x)=0$ $\begin{cases} f''(x)>0 & 极小 \\ f''(x)<0 & 极大 \\ f''(x)=0 & 无法判断 \end{cases}$

此时继续寻找 $n>1$ 的最小 n $f^{(n)}(x) \neq 0$. 若 n 为偶 $\begin{cases} f^{(n)}(x)>0 & 极小 \\ f^{(n)}(x)<0 & 极大 \end{cases}$

Mean-Value theorem 积分中值定理

f 是定义在闭区间 $[a,b]$ 的连续可导函数. 则 (a,b) 至少有一点 c

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$\left.\begin{array}{l} \text{local minimum} \\ \text{local maximum} \end{array}\right\} \Rightarrow$ extremum

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Indefinite Integration 不定积分. (antidifferentiation)

不定积分后带有常数 C.

① $\int x^n dx = \begin{cases} \frac{1}{n+1} x^{n+1} + C & \text{if } n \neq -1 \\ \log|x| + C & \text{if } n = -1 \end{cases}$

② $\int a^u du = \frac{1}{\ln a} a^u + C$ (if $a > 0$)

③ $\int \sec^2 u\, du = \tan u + C$

④ $\int \sec u \tan u\, du = \sec^2 u + C$

⑤ $\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$

Integration by parts 分部积分

$d(uv) = u\,dv + v\,du$

$\int u\,dv = uv - \int v\,du$

Trig substitutions 三角代换.

△ $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-a^2}$

$u = a\sin\theta$ $u = a\tan\theta$. $u = a\sec\theta$

Partial Fractions 因式分解.

$P(x)/Q(x)$ 因式分解.

$$\frac{1}{x^2(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}$$

$\int_a^b f(x)dx = F(b) - F(a)$.

① $\int_b^a f(x)dx = -\int_a^b f(x)dx$

② $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

③ $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

画图找出积分区域

特别的 $\frac{d}{dx}\int_{a(x)}^{b(x)} f(x)\,dx = f((b(x)))\,b'(x) - f((a(x)))\,a'(x)$

由中值定理 $\int_a^b f(x)\,dx = f(c)(b-a)$ $c \in (a,b)$

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Polar coordinates（极坐标）

$x = r\cos\theta \quad y = r\sin\theta . \quad r = \sqrt{x^2+y^2} . \quad \tan\theta = \frac{y}{x}$

在极坐标中对区域积分

$$\boxed{\int_\alpha^\beta \frac{1}{2} r^2 d\theta}$$

eg: Find the area enclosed by the cardioid $r = 2a(1+\cos\theta)$

$$S = \int_0^{2\pi} \frac{1}{2} \cdot 4a^2 (1+\cos\theta)^2 d\theta$$
$$= \int_0^{2\pi} 2a^2 (1+2\cos\theta + \cos^2\theta) d\theta$$
$$= \int_0^{2\pi} 2a^2 (1+2\cos\theta) d\theta + \int_0^{2\pi} a^2 (2\cos^2\theta - 1) d\theta + \int_0^{2\pi} a^2 d\theta .$$
$$= \int_0^{2\pi} 2a^2 d\theta + \int_0^{2\pi} a^2 d\theta$$
$$= 6\pi a^2$$

Volumes of solids of revolution 旋转体体积

$V = \int_a^b dV = \int_a^b \pi [f(x)]^2 dx$ 绕x轴

$\boxed{V = \int_a^b dv = \int_a^b \pi [g(y)]^2 dy \text{ 绕y轴.}}$

eg: The origin bounded by the curve $x = y^2+3$ and $x = 4y$ is revolved around the y-axis

$y^2+3 = 4y \quad y = 1, 3.$

$\int_1^3 \pi (\underset{f(y)}{4y})^2 - \pi (\underset{g(y)}{y^2+3})^2 dy$ 而非 $\int_1^3 \pi [4y - (y^2+3)]^2 dy.$

ARC Length 曲线长度

$$\boxed{\int ds = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.}$$

△ 重要极限 $\boxed{e \approx (1+h)^{\frac{1}{h}}}$ $h \to 0.$

eg: What's the derivative of $f(x) = x^{\sqrt{x}}$

$f(x) = e^{\sqrt{x}\log x} \quad \therefore f'(x) = e^{\sqrt{x}\log x} \left(\frac{1}{2\sqrt{x}} \log x + \frac{\sqrt{x}}{x}\right)$
$= x^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \ln x + \frac{\sqrt{x}}{x}\right)$

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Improper integrals 广义积分

unbounded intervals of the form $[a,\infty)$, $(-\infty,b]$ or $(-\infty,-\infty)$.

$\int_0^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$

第一类广义积分：区间无限.

第二类广义积分：区间内某点值无限.

两个重要极限. $\lim_{n\to 0}(1+n)^{\frac{1}{n}} = e$.

$\lim_{x\to 0}\frac{\sin x}{x} = 1$

Infinite series 数列.

$\sum\frac{1}{n^p}$ $p>1$时 收敛 $p\le 1$ 时发散.

收敛判断方法.

① $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = L < 1$.

② $\lim_{n\to\infty}\sqrt[n]{a_n} = L < 1$

Alternate series 交错级数.

$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$.

条件收敛：① $a_{n+1} < a_n$

② $a_n \to 0$ $(n\to\infty)$

若数列绝对收敛，则其一定收敛.

Power Series 幂级数.

$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = |x|\cdot\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

求出 $|x|$ 的范围.

运用此方法，需验证 $|x|$ 端点的收敛性

对于可收敛的幂级数，可以用逐项积分 求导. 若要还原，一定注意

Taylor series 泰勒展开

$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$.

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$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$ C is between 0 and x.

可用求误差·如求三阶泰勒近似误差.

$$\sum_{k=1}^{n} k = \frac{1}{2}n(1+n) \qquad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n \qquad -1 < x < 1$$

$$\frac{1}{1+x} = 1 + (-x) + x^2 + \cdots + (-1)^n x^n \qquad -1 < x < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \qquad x \in (-\infty, +\infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1}x^n}{n} \qquad -1 < x \leq 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{dy}{dx} = \frac{1-2xy}{x^2+3y^2+1}$$

$$(x^2+3y^2+1)\,dy + (2xy-1)\,\cancel{dy}\,dx = 0$$

$$\frac{\partial M}{\partial x} = 2x \quad = \frac{\partial N}{\partial y} = 2x$$

$$\int 2xy-1\,dx = x^2y - x$$

$$\cancel{xy^2 - x + g(y)} \qquad x^2 - 1 + 3y^2 + 1$$

$$\cancel{2xy} \qquad x^2y - x + y^3 + y = C$$

$$1 - 1 + 1 + 1 = 2$$

$$u + x\frac{du}{dx} = 1 + u \qquad \frac{y}{x} = \ln x + C = \ln Cx$$

$$du = \frac{1}{x}dx$$

$$C\ y = x\ln Cx$$

$$e^{\frac{y}{x}} = Cx$$

$$g(x,y) = \int x\cos xy\,dy$$

$$= \int \frac{x}{y}\,d\sin xy$$

$$= \frac{x}{y}\sin xy - \int \sin xy \cdot \frac{1}{y}\,dx$$

$$g(0,y) = \frac{x}{y}\sin xy - \frac{1}{y^2}\cos xy + \frac{1}{y^2}$$

$$-\frac{1}{y^2} + C$$

$$x\sin x - \cos x + 1$$

$2x^3+7x^2+3x=0$

$-\frac{1}{2}, -3$

$y' = C_1e^x - C_2e^{-x}$

$y = C_1e^x + C_2e^{-x}$

$\begin{cases} C_1+C_2=0 \\ C_1-C_2=0 \end{cases}$

$M = 3x^5y^4 - 5x^4y^3\,dx$

$N = 2x^6y^3 - 3x^5y^2\,dy.$

$\frac{\partial M}{\partial v}$

$\frac{1}{2}x^6y^4 - x^5y^3 + \cancel{f(y)} = C$

$y'' = p.$

$p^2-1=0 \quad p=\pm 1$

$p'' = p.$

$y'' = C_1e^x + C_2e^{-x}$

$e^{2\ln x}$

$f(x) = C_1e^x - C_2e^{-x}$

$f'(x) = C_1e^x + C_2e^{-x}$

$\int -x\,d\cos x$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x$

$15n = 10m - 10$

$15n = 9m - 6.$

$\begin{cases} m=4 \\ n=2 \end{cases}$

$3n = 2\times 4 - 2$

$2x^6y^3 - 3x^5y^2$

$\underbrace{1-\frac{1}{2}(e^x+e^{-x})}$

$1+\frac{1}{2}(e^x+e^{-x})$

$x^6y^4 - 2x^5y^3 = C$

$x^5y^3(xy-2)$

$e^{-2\ln x}\left[\int x^2\cdot\frac{\sin x}{x}\,dx + C\right]$

$\frac{1}{\frac{1}{2}\pi}[1]$

$= \frac{2}{\pi}$

$\frac{y}{x} \quad u = \frac{1}{x^2}\left[\int x\sin x\,dx + C\right]$

$\frac{1}{\frac{1}{4}\pi^2}\left[-\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+\pi^2-\pi\right]$

$\frac{1}{\frac{1}{2}\pi}\left[-\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right]$

$\frac{2}{\pi}\left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}\pi}{8}\right]$

$5n+5 = 3m+3$

$3n+6 = 2m+4$

$\frac{1}{x^2}\left[-x\cos x + \sin x + C\right]$

$y = \frac{1}{x}\left[-x\cos x + \sin x + C\right]$

$\underbrace{\frac{\sqrt{2}}{\pi} - \frac{1}{2\sqrt{2}}}$

$\begin{cases} 3n = 2m-2 \\ 5n = 3m-2 \end{cases}$

$\frac{1}{\pi^2}[\pi + C] = 1$

$C = \pi^2 - \pi$

$\frac{1}{\pi}[+\pi + C] = 1 \quad C = 2\pi$

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$y'' = -2a^2\cancel{\cos ax\, e^{ax}} = a^2$

$\cancel{-2a^2 \sin ax\, e^{ax}}$

$-2a^2 = a^2$

$\frac{1}{2}(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \frac{1}{7})$

$\frac{1}{2}$

$y = (\cos ax - \sin ax)e^{ax}$

$y' = -a\sin ax - \cancel{a\cos ax} + \cancel{a\cos ax} - a\sin ax$

$= -2a\sin ax\, e^{ax}$

$y' = (-a\sin ax + aC_2\cos ax)e^{ax} + (a\cos ax + aC_2\sin ax)e^{ax} = 0$

$y(0)$

$\frac{1}{x^2-y^2} = cx^2$

$x^2 - y^2 = c$

$y = (\cos ax - \sin ax)e^{ax}$ $\quad C_2 = -1$

$x^3 - ax^2 + a^2x - a^3 = 0$

$c_1 e^0 = 1$

$-\ln(1-u^2) = \ln x^2 c$

$x^2(x-a) + a^2(x-a) = 0$

$x \cdot \frac{du}{dx} = \frac{1-u^2}{u}$

$\frac{1}{1-u^2} = cx^2$

$(x^2+a^2)(x-a) = 0$

$\frac{u}{1-u^2}du = \frac{1}{x}dx$

$x^2y^2 + 2xuy + u^2 = 2 + 2\frac{x}{u}y - \frac{1}{u^2}$

$\frac{1}{1-\frac{y^2}{x^2}} = cx^2$

$\frac{-2u}{1-u^2}$

$x^2y^2 + 2(xu - \frac{x}{u})y + u^2 + \frac{1}{u^2} - 2 = 0$

$xy = \frac{1}{u} - u$

$(2a^2 + 2ar\sin\theta)^{\frac{1}{2}} \times 2$

$x^2y^2 + 2x(u - \frac{1}{u})y + (u - \frac{1}{u})^2 = 0.$

$\int_{\frac{\pi}{3}}^{\frac{3\pi}{3}} \int_0^a r\sqrt{2a(a + r\sin\theta)}\, dr\, d\theta$

$\frac{2}{3}(2a^2 + 2ar\sin\theta)^{\frac{3}{2}}$

$(xy + u - \frac{1}{u})^2 = 0$

$-\frac{1}{2}\ln(1-u^2) = \ln x + c$

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1 2 3 4 5 6 7 8 9

4 6 9 7 2 5 8 1 3

4 6 9 7 2 5 8 1 3
1 2 3 4 5 6 7 8 9

(1, 8, 7, 4)(2, 5, 6)

$e^{-\sin x} < e^0$

$-\sin x < 0$

$\sin x > 0$

$0, \pi$

$2n\pi,\ (2n+1)\pi$

$\frac{e^0}{n^2} = \frac{1}{n^2}$

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$2a^2\sin\frac{\theta}{2}\Big|_0^\pi$

$\int_0^\pi a^2\cos\frac{\theta}{2}\,d\theta$

$2ay - y^2 + y^2$

$x^2+y^2=2ay$

$2\times\frac{2}{3}a^3\sin\frac{\theta}{2}$

$\frac{4}{3}a^3$.

$x^2+y^2-2ay+a^2=a^2$

$x^2+(y-a)^2=a^2$

$y=a+a\sin\theta$

$x=a\cos\theta$

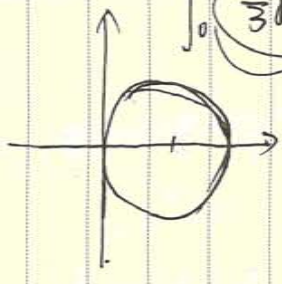
$2a(a+a\cos\theta)$

$=2a^2(1+\cos\theta) = 2a^2(1+2\cos^2\frac{\theta}{2}-1)=$

$\int_0^\pi\int_0^a 2\cos\frac{\theta}{2}\cdot r^2\,dr\,d\theta$.

$\int_0^\pi r^2\cos\frac{\theta}{2}\Big|_0^a\,d\theta$.

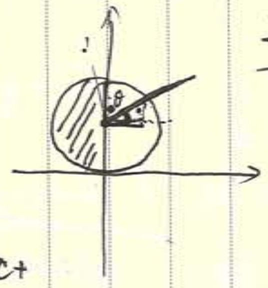
$\int_0^\pi \frac{2}{3}a^3\cos\frac{\theta}{2}\,d\theta$



$a^2(2+2\cos\theta)$

$2(2\cos^2\frac{\theta}{2}-1)+2$

$\frac{4\cos^2\frac{\theta}{2}\cdot a^2}{2\cos\frac{\theta}{2}\cdot r^2}$



$x^2+y^2=2a\cdot(a+a\sin\theta)$

$=a^2(2+2\sin\theta)$

$2a(a+r\cos\theta)$

$(\sin^3t\cos xy-1)\cdot 3\cos^2t(-\sin t) + (1+\cos^3t\cos xy)\cdot 3\sin^2t\cos t\;dt$

$-3\cos^2t\sin^4t\cos xy + 3\cos^2t\sin t + 3\sin^2t\cos t + 3\sin^2t\cos^4t\cos xy$

$3\sin^2t\cos^2t\cos xy(\cos^2t-\sin^2t) + 3\sin t\cos t(\sin t+\cos t)$

$\frac{3}{4}\sin^2 2t\cos 2t\cos xy + \frac{3}{2}\sin 2t(\sin t+\cos t)\;dt$

$(y\cos xy)'$

$(1+x\cos xy)'$

$-3\cos^2 t\sin t\ \ \sin^3 t$

$\cos xy \cdot yx\sin xy$

$-(\cos xy - xy\sin xy)$

$\cos^3 t\sin^3 t$

$= \frac{1}{8}\sin^3 2t$

$\cos xy + yx(-\sin xy)$

$-\cos xy$.

$y = x^{\frac{3}{2}}$

$y' = \frac{3}{2}x^{\frac{1}{2}}$

$\sqrt{1+\frac{9}{4}x}$

$\int (x^3 - y^2 + 27)\, ds$

$\int_0^1 (x^3 - x^3 + 27)\sqrt{1+\frac{9}{4}x}\, dx$.

$\int_0^1 \frac{27}{2}\sqrt{4+9x}\, dx$.

$(4+9x)^{\frac{1}{2}}$

$(4+9x)^{\frac{3}{2}}$

$\int_0^1 \frac{1}{27}\, d\sqrt{4+9x}^{\,3}$

$\frac{3}{2}(4+9x)^{\frac{1}{2}}\cdot 9$

$\frac{27}{2}$

$x^3 - x^2 + x^4 \Big|_{-1}^{0}$

$\frac{1}{27}\cdot(4+9x)^{\frac{3}{2}}\Big|_0^1$

$\frac{1}{27}$.

$-1-1+\frac{1}{2}$

$4+$

$13^{\frac{3}{2}} - 4^{\frac{3}{2}}$

$13\sqrt{\ }$ 8

$-\frac{3}{2}$

$\frac{3}{4}$

$= 1$

$\int (3y-2x)dx + (x^2+y)dy$.

$-1-1+1$

$(3y-2x) + (x^2+x^2)\cdot 2x\ dx$

$\int_{-1}^{0} (3x^2 - 2x + 4x^3)\, dx$

$x-z=a$
$y-z=b$ $\Rightarrow$ $x-y=a-b$
$x-2y=c$ $y=a-b-c$

$x=a-b+a-b-c$
$x=2a-2b-c$
$z=a-2b-c$

1. 由机械泵与扩散泵组成的真空排气系统，设扩散泵的抽速为4000 L/s，进口压强为0.1Pa，出气口的压强为10Pa，问选择抽速多大的机械泵与其匹配

2. 为什么热偶真空不能用于较低或较高真空度的测量 只能用于100-0.1Pa

$\cancel{2a-2b-c-a+b+c+3a-6b}-3c$
$4a-7b-3c$

$5x^2z^4z'-2yz^3-3y^2z^2z'-3x=0$
$-5z'\cdot2-3z'+3=0$

$-z^5-z^3+3=1$
$2=z^5+z^3$ 1=
$z=1$

$\cancel{2yz+y^2z'-2z^2-4xz}$
$2yz+y^2z_y'-4xzz_y'+3x^2=0$
$2+zy'-4zy'+3=0$ $5=3zy'$
$y^2z_x'-2z^2-4xzz_x'+3y\cdot2x=0$

$5=8z'$
$z'=\frac{5}{8}$

$z_x'=\frac{4}{3}$ $z_y'=\frac{5}{3}$

$z_x'-2-4z_x'+6=0$
$4=3z_x'$

$z-1=\frac{4}{3}(x-1)+\frac{5}{3}(y-1)$
$3z-3=4x-4+5y-5$
$3z-3=4x+5y-9$
$6=4x+5y-3z$

$4\times6-20-9$ $4\times3-5-15$
$24+15-$ $-8+20-6=$

$6x^2-4x^2=20$ -1
$2x^2=20$
$\sqrt{20}$

$10x^2=20$
$x^2=2$
$2x^2=4$

$x=3\lambda x+2\lambda y$ $(1-3\lambda)x=2\lambda y$
$y=2\lambda x+3\lambda y$ $\frac{x}{y}=\frac{2\lambda}{1-3\lambda}$
$y(1-3\lambda)=2\lambda x$ $\frac{x}{y}=\frac{1-3\lambda}{2\lambda}=\frac{1-3}{2}=-1=\frac{1-\frac{3}{5}}{\frac{2}{5}}=1$

$\lambda=1$ $\lambda=\frac{1}{5}$

$(2\lambda)^2=(1-3\lambda)^2$ $5\lambda^2-6\lambda+1=0$
$4\lambda^2=9\lambda^2-6\lambda+1$ $(5\lambda-1)(\lambda-1)=0$